Convex Functions: Part 3

Lecture 7-3 - CMSE 382

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This Lecture

Topics:

- Continuity and differentiability of convex functions
- Extended real-valued functions
- Maxima of a convex function

Announcements:

- Homework 3 due Friday!
- The homework uses CVXPY. If you missed last class, make sure you get your CVXPY installation working ASAP!

Section 1

Continuity and differentiability of convex functions

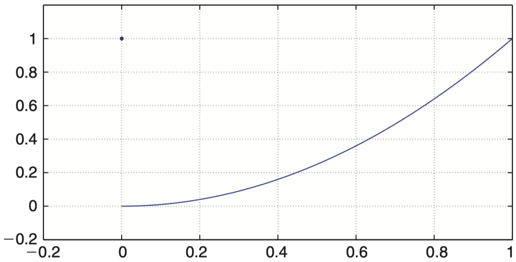
Are convex functions always continuous?

Definition

A function $f : C \to \mathbb{R}$ defined over a convex set $C \subseteq \mathbb{R}^n$ is **convex** if for any $\mathbf{x}, \mathbf{y} \in C$ and $\lambda \in [0,1]$, we have

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}).$$

Nope.



Lipschitz continuity

Definition (Lipschitz Continuity)

A function $f\colon S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$ is **Lipschitz continuous** if there exists an $L > 0$ such that $\|f(\mathbf{x}) - f(\mathbf{y})\| \leq L\|\mathbf{x} - \mathbf{y}\| \; \forall \mathbf{x}, \mathbf{y} \in S$.

- Note that f isn't assumed to be continuous.
- Lipschitz continuity implies continuity, but the converse is not necessarily true.

Local Lipschitz Continuity

Definition (Local Lipschitz Continuity)

A function $f : S \to \mathbb{R}$ where $S \subseteq \mathbb{R}^n$ is **locally Lipschitz continuous** if for every $\mathbf{x}_0 \in S$ there exist $\varepsilon > 0$ and $L > 0$ such that $B(\mathbf{x}_0, \varepsilon) \subseteq S$ and $|f(\mathbf{x}) - f(\mathbf{x}_0)| \leq L\|\mathbf{x} - \mathbf{x}_0\|, \ \forall \mathbf{x} \in B(\mathbf{x}_0, \varepsilon)$.

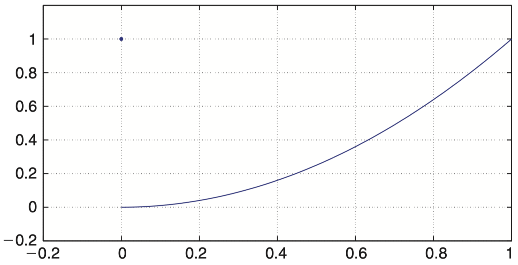
Continuity of convex functions

Are convex functions always continuous?

Theorem (Local Lipschitz continuity of convex functions)

Let $f : C \to \mathbb{R}$ be a convex function defined over a convex set $C \subseteq \mathbb{R}^n$. Then f is locally Lipschitz continuous at every $\mathbf{x}_0 \in int(C)$.

- Convex functions on $C \subseteq \mathbb{R}^n$ are continuous on $\text{int}(C)$.
- Convex functions on $\mathbb{R}^n$ are continous on $\mathbb{R}^n$.



Directional derivatives for convex functions

Definition (Recall)

The **directional derivative** of $f : \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{x}$ along the direction $\mathbf{d}$ is defined as $f'(\mathbf{x}; \mathbf{d}) = \nabla f(\mathbf{x})^\top \mathbf{d}$.

Theorem (Existence of directional derivatives for convex functions)

Let $f : C \to \mathbb{R}$ be a convex function defined over the convex set $C \subseteq \mathbb{R}^n$. Let $\mathbf{x} \in int(C)$. Then for any $\mathbf{d} \neq \mathbf{0}$, the directional derivative $f'(\mathbf{x}; \mathbf{d})$ exists.

Section 2

Extended real-valued functions

Extended real-valued functions

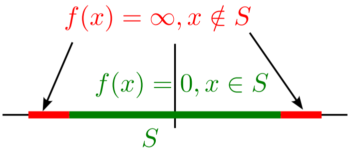
Definition (Extended real-valued functions)

- Real-valued functions take their values in $\mathbb{R} = (-\infty, \infty)$.
- Extended real-valued functions take their values in $\mathbb{R} \cup \{\infty\} = (-\infty, \infty]$

- **Warning:** Some sources define the extended real numbers as $\mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$.

Example: Indicator function

$$\delta_S(\mathbf{x}) = \begin{cases} 0, & \text{if } \mathbf{x} \in S \\ \infty, & \text{if } \mathbf{x} \notin S \end{cases}$$



Arithmetic with ∞:

- $a + \infty = \infty$ for any $a \in \mathbb{R}$.
- $a \cdot \infty = \infty$ for any $a \in \mathbb{R}_{++}$.
- $0 \cdot \infty = 0$.

Effective domain of extended real-valued functions

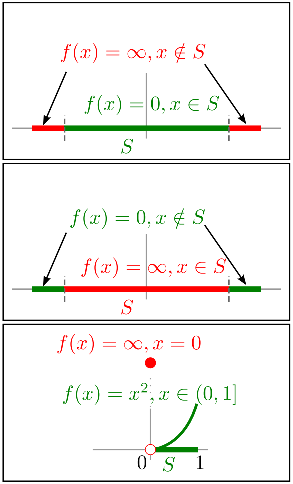
Definition

The **effective domain of an extended real-valued function** is the set of vectors for which the function takes a real value:
$\text{dom}(f) = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) < \infty\}$.

$$\delta_S(x) = \begin{cases} 0, & \text{if } \mathbf{x} \in S; \\ \infty, & \text{if } \mathbf{x} \notin S. \end{cases}$$

$$f(x) = \begin{cases} \infty, & \text{if } \mathbf{x} \in S; \\ 0, & \text{if } \mathbf{x} \notin S. \end{cases}$$

$$f(x) = \begin{cases} \infty, & \text{if } x = 0; \\ x^2, & \text{if } x \in (0,1]. \end{cases}$$

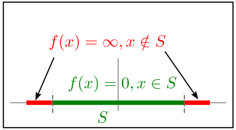


Convexity of extended real-valued functions

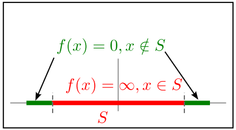
Theorem

An extended real-valued function f is convex if and only if $dom(f)$ is a convex set and f is convex over $dom(f)$.

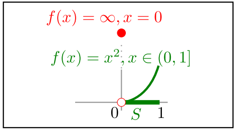
$$\delta_S(x) = \begin{cases} 0, & \text{if } \mathbf{x} \in S; \\ \infty, & \text{if } \mathbf{x} \notin S. \end{cases}$$



$$f(x) = \begin{cases} \infty, & \text{if } \mathbf{x} \in S; \\ 0, & \text{if } \mathbf{x} \notin S. \end{cases}$$



$$f(x) = \begin{cases} \infty, & \text{if } x = 0; \\ x^2, & \text{if } x \in (0, 1]. \end{cases}$$



Epigraph of a function

Definition (Epigraph of a function)

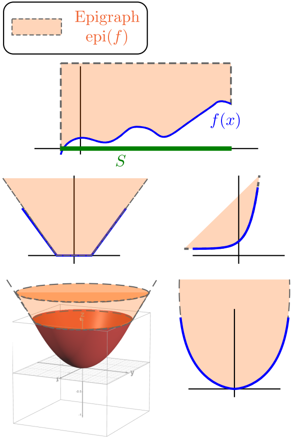
Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$. The epigraph set $\text{epi}(f) \subseteq \mathbb{R}^{n+1}$ is defined by

$$\text{epi}(f) = \left\{ \begin{pmatrix} \mathbf{x} \\ t \end{pmatrix} : f(\mathbf{x}) \leq t \} \right\}.$$

- It contains all the points $\begin{pmatrix} \mathbf{x} \\ t \end{pmatrix}$ on or above the function graph.

Theorem (Function and epigraph convexity)

An extended real-valued (or a real-valued) function f is convex if and only if its epigraph set $\text{epi}(f)$ is convex.



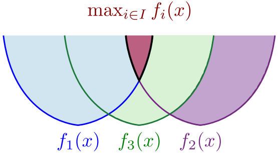
Preservation of convexity under maximum for extended real-valued functions

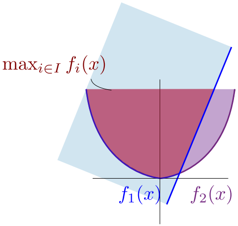
Theorem

Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be an extended real-valued convex function for any $i \in I$ (I being an arbitrary index set). Then the function

$$f(\mathbf{x}) = \max_{i \in I} f_i(\mathbf{x})$$

is an extended real-valued convex function.





Section 3

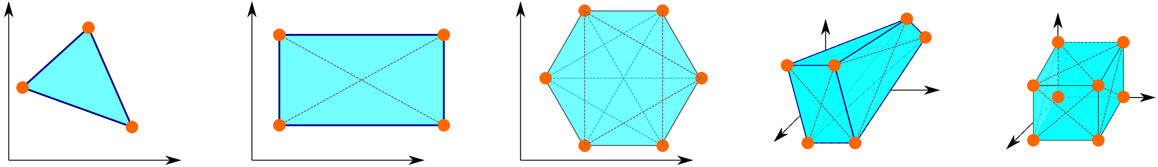
Maxima of a convex function

Recall: Extreme points

Definition (Recall: Extreme points)

Let S be a convex set. A point $\mathbf{x} \in S$ is an **extreme point** of S if there do not exist two distinct points $\mathbf{x}_1, \mathbf{x}_2 \in S$ and $\lambda \in (0,1)$ such that $\mathbf{x} = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$.

- It is a point in S that cannot be represented as a nontrivial convex combination of two different points in S.
- The set of all extreme points is denoted $\text{ext}(S)$.

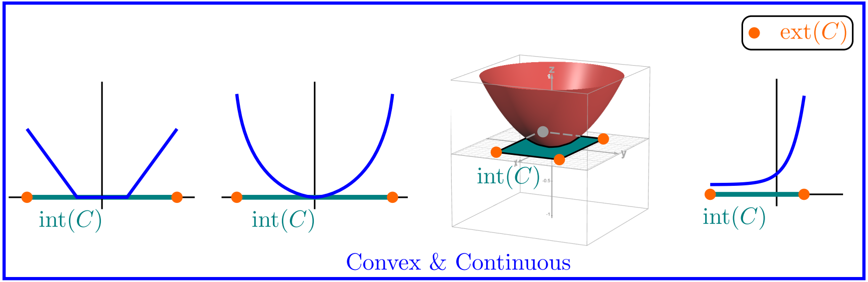


Maxima of convex functions

Theorem

Let $f : C \to \mathbb{R}$ be a convex function which is not constant over the convex set C. Then f does not attain a maximum at a point in int(C).

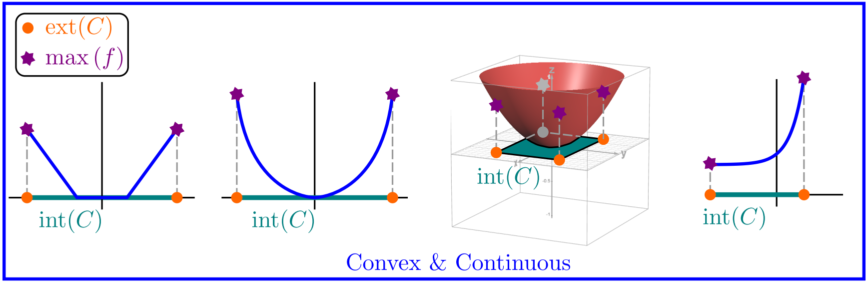
- *Maximum of non-constant convex function defined on a convex set cannot occur at an interior point in the set.*



Maxima of convex functions

Theorem

Let $f : C \to \mathbb{R}$ be a convex and continuous function over the convex and compact set $C \subseteq \mathbb{R}^n$. Then there exists at least one maximizer of f over C that is an extreme point of C.



Groups - Round 4

Group 1
Michal, Joseph, Saitej, Dev

Group 2
Kyle, Dori, Shanze, Jack

Group 3
Noah, Daniel, Lora, Scott

Group 4
Lowell, Tianjian, Aidan, Anthony

Group 5
Abigail, Breena, Arjun, Luis

Group 6
Purvi, Atticus, Andrew, Vinod

Group 7
Yousif, Jay, Arya, Morgan

Group 8
Jonid, Jake, Dominic, Maye

Group 9
Alice, K M Tausif, Monirul Amin, Ha

Group 10
Jamie, Zheng, Aaron, Long

Group 11
Lauryn, Karen, Sanskaar, Braedon

Group 12
Sai, Brandon, Igor, Quang Minh