Optimizality Conditions: Part 1

Lecture 2-1 - CMSE 382

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::
Dept of Computational Mathematics, Science & Engineering

Weds, Jan 21, 2026

Topics covered

- Global optima
- Extreme value theorem
- Stationary points
- First order optimality condition

Section 1

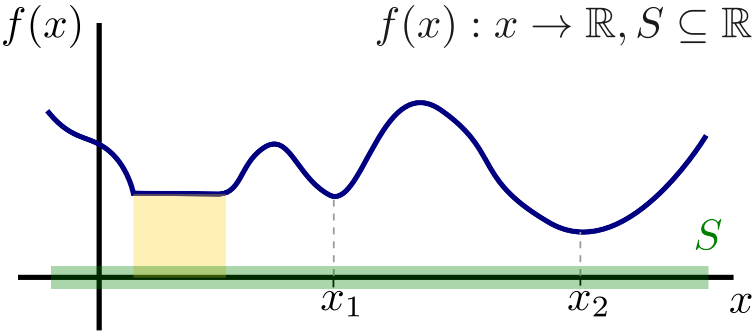
Global and local optima

Local minima

Definition

Let $f : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}^n$, then

- x^* is a **local minimum** over S if $\exists r > 0$ such that $f(x) \geq f(x^*) \forall x \in S \cap B(x^*, r)$.
- x^* is a **strict local minimum** over S if $\exists r > 0$ such that $f(x) > f(x^*) \forall x \in S \cap B(x^*, r)$.

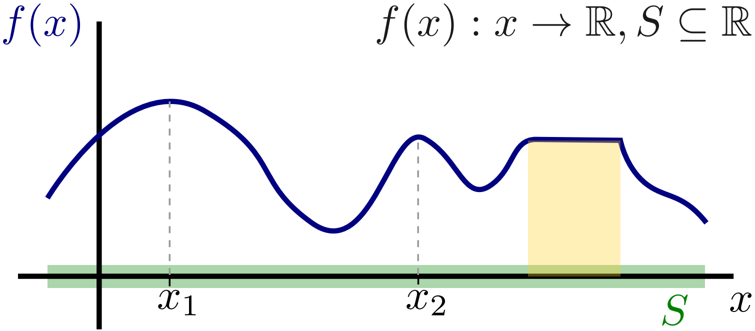


Local maxima

Definition

Let $f : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}^n$, then

- x^* is a **local maximum** over S if $\exists r > 0$ such that $f(x) \leq f(x^*) \forall x \in S \cap B(x^*, r)$.
- x^* is a **strict local maximum** over S if $\exists r > 0$ such that $f(x) < f(x^*) \forall x \in S \cap B(x^*, r)$.

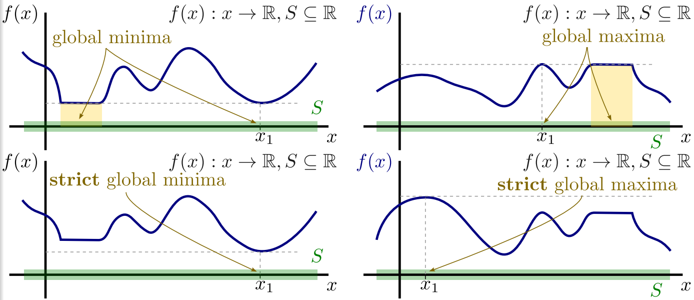


Global optima

Definition

Let $f : S \to \mathbb{R}$, where $S \subseteq \mathbb{R}^n$, then $x^* \in S$ is a

- **global minimum** over S if $f(x) \geq f(x^*)\ \forall x \in S$.
- **strict global minimum** over S if $f(x) > f(x^*)\ \forall x \in S$.
- **global maximum** over S if $f(x) \leq f(x^*)\ \forall x \in S$.
- **strict global maximum** over S if $f(x) < f(x^*)\ \forall x \in S$.



Generalized extreme value theorem

Theorem (Generalized Extreme value theorem (EVT))

Let $U \subset \mathbb{R}^n$ be a non-empty **compact** *set. If $f : U \to \mathbb{R}$ is a continuous function, then f is* **bounded** *and there exists $p, q \in U$ such that $f(p) = \sup_{x \in U} f(x)$ and $f(q) = \inf_{x \in U} f(x)$.*

Recall

- A subset of $\mathbb{R}^n$ is **compact** if it is **closed** and **bounded**.
  - A set U is **bounded** if there exist $M > 0$ such that $U \subset B(\mathbf{0}, M)$, i.e., we can contain the set within a finite open ball.
  - A set U is **closed** with respect to a metric d if it contains all its limits points, i.e., $U = \text{Cl}(U)$

Critical points

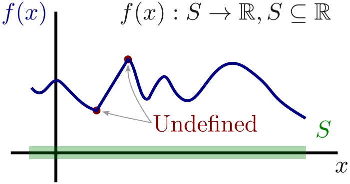
Definition (Critical points)

Let $f : U \to \mathbb{R}$, a point $x^* \in U$ is called a **critical point** of f if $\nabla f(x^*) = 0$, or if $\nabla f(x^*)$ is not defined.

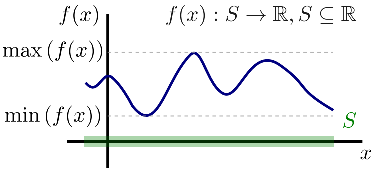
A critical point can correspond to:

- $\nabla f(x^*)$ undefined (including boundary points of U)
- local minimum ($\nabla f(x^*) = 0$)
- local maximum ($\nabla f(x^*) = 0$)
- saddle point ($\nabla f(x^*) = 0$)

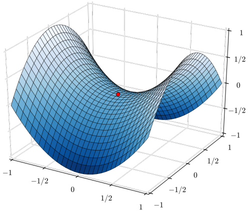
undefined derivative



Max/Min



Saddle



$z = x^2 - y^2$

Stationary points

Definition (Stationary points)

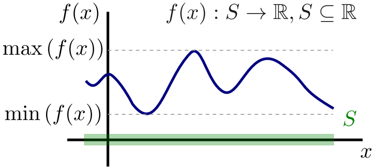
Let $f : U \to \mathbb{R}$, $x^* \in \text{int}(U)$, and f be differentiable over a neighborhood of x^*, then x^* is called a **stationary point** of f if $\nabla f(x^*) = 0$.

A stationary point can correspond to:

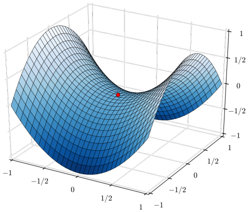
- local minimum ($\nabla f(x^*) = 0$)
- local maximum ($\nabla f(x^*) = 0$)
- saddle point ($\nabla f(x^*) = 0$)

Warning: *Stationary points are critical points, but critical points are not necessarily stationary points.*

Max/Min



Saddle



$z = x^2 - y^2$

First order optimality condition

Theorem (First order optimality condition)

Let $f : U \to \mathbb{R}$ be a function defined on $U \subseteq \mathbb{R}^n$. Suppose $x^* \in int(U)$ is a local optimum (minimum or maximum) point and that all partial derivatives of f exist at x^*.
Then $\nabla f(x^*) = 0$.

Group Work Time

- Find and download the worksheet from the course website.
- At the end of class, you will upload your completed worksheet to D2L. Note: graded on completion only.
- The last 15 minutes will be for the quiz.

Groups - Round 1

Group 1
Jonid M.
Sanskaar M.
Andrew B.
Zheng Y.

Group 2
Abigail P.
Braedon P.
Vinod R.
Aidan S.

Group 3
Luis C.
Tianjian X.
Lowell M.
Arjun R.

Group 4
Jay B.
Maye B.
Jamie L.
Kyle S.

Group 5
Aaron N.
Purvi G.
Morgan F.
Breena K.

Group 6
Brandon G.
Dominic V.
K M Tausif S.
Anthony K.

Group 7
M.A. Mahin
Dan E.
Quang Minh D.
Ha N.

Group 8
Lora S.
Jack C.
Noah M.
Michal T.

Group 9
Atticus B.
Shanze O.
Joseph M.
Arya S.

Group 10
Scott W.
Karen S.
Dev A.
Dori C.

Group 11
Saitej B.
Jake R.
Alice S.
Long N.

Group 12
Lauryn C.
Sai P.
Yousif E.
Igor A.J.

Next time

- Check course webpage for videos and reading for next class
- Office hours will be scheduled for next week

- Quiz 1 on Wednesday, Jan 21.
- Cheat sheet allowed.
  - One 8.5 x 11 inch sheet of paper, front and back.
  - Handwritten only.
  - Cannot be duplicate/photocopy/etc of someone else's work.
  - You will turn this in with your quiz.
  - Failure to follow these rules will result in a 10% deduction on the quiz.
- Calculator allowed (no internet)
- Note: Lowest grade dropped at end of semester.