The KKT Conditions

Lecture 11-2 - CMSE 382

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This Lecture

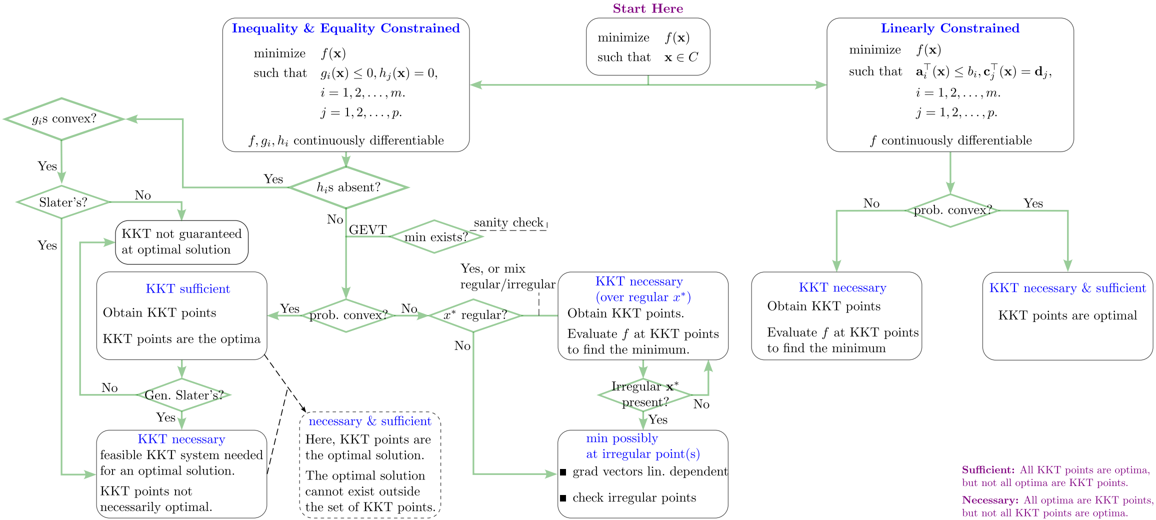
Topics:

- The convex case: KKT sufficiency
- Slater conditions
- The convex case: KKT necessity

Announcements:

- HW...

Big picture view



Sufficient: All KKT points are optima, but not all optima are KKT points.

Necessary: All optima are KKT points, but not all KKT points are optima.

Sufficiency and Necessity

Sufficient

- All KKT points are optima.
- Optima might not be KKT points.

Necessary

- All optima are KKT points.
- KKT points might not be optima.

Section 1

Convex case: KKT sufficiency

Recall: KKT conditions for Inequality and equality constrained problems

Theorem (Recall: Inequality and equality constrained problems)

Let $\mathbf{x}^*$ be a local minimum of the problem

$$\begin{aligned} \textit{min} \quad & f(\mathbf{x}) \\ \textit{such that} \quad & g_i(\mathbf{x}) \leq 0, i = 1, 2, \ldots m, \\ & h_j(\mathbf{x}) = 0, j = 1, 2, \ldots p. \end{aligned}$$

where $f, g_1, \ldots, g_m, h_1, h_2, \ldots, h_p$ are continuously differentiable functions over $\mathbb{R}^n$. Suppose that $\mathbf{x}^*$ is **regular**, *then $\mathbf{x}^*$ is a* **KKT point**.

- A necessary condition for local optimality of a regular point is that it is a KKT point.
- Regularity is not required in the linearly constrained case.

Sufficiency of KKT conditions for **convex** problems

Theorem

Let $\mathbf{x}^*$ *be a feasible solution of*

$$\begin{aligned} \text{min} \quad & f(\mathbf{x}) \\ \text{such that} \quad & g_i(\mathbf{x}) \leq 0, i = 1, \ldots m, \\ & h_j(\mathbf{x}) = 0, j = 1, \ldots p. \end{aligned}$$

where $f, g_1, \ldots, g_m$ *are continuously differentiable convex functions over* $\mathbb{R}^n$*, and* $h_1, h_2, \ldots, h_p$ *are affine functions.*

Suppose that there exist $\lambda_1, \ldots, \lambda_m \geq 0$ *and* $\mu_1, \ldots, \mu_p \in \mathbb{R}$ *such that*

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(\mathbf{x}^*) = \mathbf{0},$$

$$\lambda_i g_i(\mathbf{x}^*) = 0, i = 1, \ldots, m.$$

Then $\mathbf{x}^*$ *is an* ***optimal solution****.*

In convex problems, the KKT conditions are always sufficient for optimality. No further conditions (such as regularity) are required.

Section 2

Convex case: KKT necessity

Slater's condition

Definition (Slater's condition)

Given a set of convex inequalities

$$g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \ldots, m,$$

where $g_1, g_2, \ldots, g_m$ are given convex functions, we say that **Slater's condition** is satisfied if there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that

$$g_i(\hat{\mathbf{x}}) < 0, \quad i = 1, 2, \ldots, m.$$

- Requires a point that strictly satisfies the constraints.
- Does not require knowledge on candidates for the optimal solution.
- Usually easier to check than regularity.

Necessity of KKT conditions under Slater's condition

Theorem

Let $\mathbf{x}^*$ *be an optimal solution of the problem*

$$\begin{aligned} &\min\ f(\mathbf{x}) \\ &s.t.\ g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \ldots, m, \end{aligned}$$

where $f, g_1, \ldots, g_m$ *are continuously differentiable functions over* $\mathbb{R}^n$*. In addition, assume* $g_1, g_2, \ldots, g_m$ *are convex functions and there exists* $\hat{\mathbf{x}}$ *which satisfies* **Slater's condition***. Then* $\mathbf{x}^*$ *is a KKT point.*

Note: The point $\hat{\mathbf{x}}$ from Slater's condition does not need to be the same as $\mathbf{x}^*$ (the candidate for the optimal solution).

Generalized Slater's condition

Definition (Generalized Slater's condition)

Consider the system

$$\begin{aligned} g_i(\mathbf{x}) &\leq 0, \quad i = 1, 2, \ldots, m, \\ h_j(\mathbf{x}) &\leq 0, \quad j = 1, 2, \ldots, p, \\ s_k(\mathbf{x}) &= 0, \quad k = 1, 2, \ldots, q, \end{aligned}$$

where $g_1, g_2, \ldots, g_m$ are convex functions, and $h_1, h_2, \ldots, h_p, s_1, s_2, \ldots, s_k$ are affine functions. We say that **generalized Slater's condition** is satisfied if there exists $\hat{\mathbf{x}} \in \mathbb{R}^n$ such that

$$\begin{aligned} g_i(\hat{\mathbf{x}}) &< 0, \quad i = 1, 2, \ldots, m, \\ h_j(\hat{\mathbf{x}}) &\leq 0, \quad j = 1, 2, \ldots, p, \\ s_k(\hat{\mathbf{x}}) &= 0, \quad k = 1, 2, \ldots, q. \end{aligned}$$

Necessity of KKT conditions under the generalized Slater's condition

Theorem

Let $\mathbf{x}^*$ *be an optimal solution of*

$$\begin{aligned} \underset{\mathbf{x}}{\text{minimize}} \quad & f(\mathbf{x}) \\ \text{subject to} \quad & g_i(\mathbf{x}) \leq 0, i = 1, \ldots, m, \\ & h_j(\mathbf{x}) \leq 0, j = 1, \ldots, p, \\ & s_k(\mathbf{x}) = 0, k = 1, \ldots, q, \end{aligned}$$

where f *and all* g_i *are continuously differentiable convex functions over* $\mathbb{R}^n$*, and all* h_j*,* s_k*, are affine. Suppose that there exists* $\hat{\mathbf{x}} \in \mathbb{R}^n$ *satisfying the* **generalized** *Slater's condition.*

Then $\mathbf{x}^*$ *is a KKT point, i.e. there exist multipliers* $\lambda_1, \ldots, \lambda_m, \eta_1, \ldots, \eta_p \geq 0$*,* $\mu_1, \ldots, \mu_q \in \mathbb{R}$ *such that*

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}^*) + \sum_{j=1}^{p} \eta_j \nabla h_j(\mathbf{x}^*) + \sum_{k=1}^{q} \mu_k \nabla s_k(\mathbf{x}^*) = 0,$$

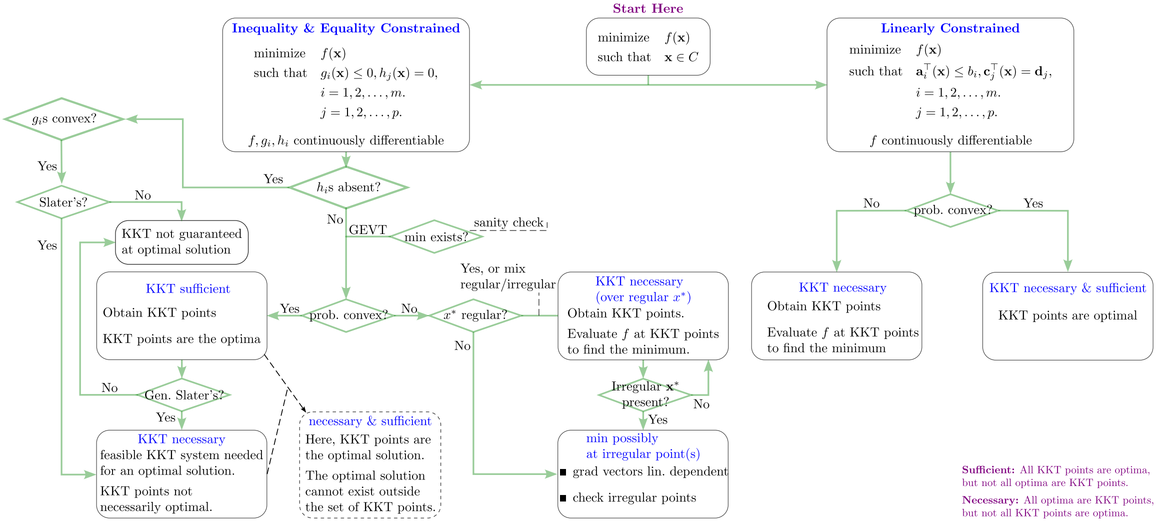
$$\lambda_i g_i(\mathbf{x}^*) = 0, \quad i = 1, \ldots, m,$$

$$\eta_j h_j(\mathbf{x}^*) = 0, \quad j = 1, \ldots, p.$$

Section 3

Big picture

Big picture view



Groups - Round 5

Group 1
Michal, Kyle, Daniel, Purvi

Group 2
Joseph, Jack, Scott, Breena

Group 3
Saitej, Dori, Noah, Tianjian

Group 4
Dev, Shanze, Lowell, Andrew

Group 5
Lora, Aidan, Arjun, Monirul Amin

Group 6
Anthony, Abigail, Atticus, Yousif

Group 7
Luis, Vinod, Morgan, Dominic

Group 8
Jay, Jonid, Alice, Aaron

Group 9
Arya, Jake, K M Tausif, Lauryn

Group 10
Maye, Ha, Zheng, Sai

Group 11
Jamie, Karen, Brandon, Quang Minh

Group 12
Long, Sanskaar, Braedon, Igor