

Name:

Present group members:

Worksheet 2-1: Q1

The function $f(x) = 3x^4 - 28x^3 + 84x^2 - 96x$ over $[-1, 6]$ is plotted at [desmos.com/calculator/nevlfz0yxp](https://www.desmos.com/calculator/nevlfz0yxp).

The derivative of f is $f'(x) = 12x^3 - 84x^2 + 168x - 96 = 12(x - 1)(x - 2)(x - 4)$.

- Using the derivative, find all critical points of $f(x)$.
 - Critical points are when $f'(x) = 0$ or is undefined, so set $12(x - 1)(x - 2)(x - 4) = 0$.
 - Critical points are $x = -1, 1, 2, 4, 6$ since we need both the points where the derivative is zero and the endpoints of the interval.

- Identify all stationary points, and for each stationary point, identify its type (min, max, saddle).
 - The critical points which were found using $f'(x) = 0$ are also stationary points, so $x = 1, 2, 4$.
 - We can check min/max/saddle type by checking the second derivative,
 $f''(x) = 36x^2 - 168x + 168$
 - $x = 1$: $f''(1) = 36 > 0$, so this is a local min.
 - $x = 2$: $f''(2) = -24 < 0$, so this is a local max.
 - $x = 4$: $f''(4) = 72 > 0$, so this is a local min.

- Find the global minimum and maximum of $f(x)$ over the interval $[-1, 6]$.
 - For 1D function, we check the critical points plus the end points:
 - $f(-1) = 211$
 - $f(1) = -37$
 - $f(2) = -32$
 - $f(4) = -64 \rightarrow$ Smallest value on the list, so global minimum is at $x = 4$
 - $f(6) = 288 \rightarrow$ Largest value on the list, so global maximum is at $x = 6$

Worksheet 2-1: Q2

Find the stationary points of $f(x, y) = 6x^2y - 3x^3 + 2y^3 - 150y$

- $\nabla f(x) = \begin{bmatrix} 12xy - 9x^2 \\ 6x^2 + 6y^2 - 150 \end{bmatrix}$
- Set each component to zero:
 - $12xy - 9x^2 = 0 \implies 3x(4y - 3x) = 0 \implies x = 0$ or $y = \frac{3}{4}x$
 - $6x^2 + 6y^2 - 150 = 0 \implies x^2 + y^2 = 25$
- Case 1: $x = 0 \implies y^2 = 25 \implies y = \pm 5$. So stationary points at $(0, 5)$ and $(0, -5)$.
- Case 2: $y = \frac{3}{4}x$. Substitute into second equation:
 - $x^2 + \left(\frac{3}{4}x\right)^2 = 25 \implies x^2 + \frac{9}{16}x^2 = 25 \implies \frac{25}{16}x^2 = 25 \implies x^2 = 16 \implies x = \pm 4$
 - So $y = \frac{3}{4}(\pm 4) = \pm 3$. So stationary points at $(4, 3)$ and $(-4, -3)$.
- Visual these points on the graph at [desmos.com/3d/xa4komuwmb](https://www.desmos.com/3d/xa4komuwmb)

Worksheet 2-1: Q3 On a quiz, Dr. Munch asks about a function $f : U \rightarrow \mathbb{R}$ defined on $U \subseteq \mathbb{R}^n$ where all partial derivatives of f exist.

- Kylo Ren writes the following on his quiz.

At a local optimum, the gradient is zero, so $\nabla f(x^*) = 0$.

Mark his answer correct or explain why his answer is wrong.

- This almost looks like the local minimum first order necessary condition, but it is missing a key detail.
- The condition only holds when x^* is in the interior of U .
- If x^* is on the boundary of U , then the gradient may not be zero.
- Example: $f(x) = x$ on $U = [0, 1]$. The global minimum is at $x^* = 0$, but $\nabla f(0) = 1 \neq 0$.

- Rey Skywalker writes the following on her quiz.

Since $\nabla f(\mathbf{x}^*) = 0$, f has a local optimum.

Mark her answer correct or explain why her answer is wrong.

- This is not necessarily true. Having a zero gradient is a necessary condition for a local optimum, but it is not sufficient. For example, the point could also be a saddle point.

- Poe Dameron writes the following on his quiz.

Since f has a local optimum at \mathbf{x}^* and \mathbf{x}^* is in the interior of U , the gradient is zero, meaning $\nabla f(\mathbf{x}^*) = 0$.

Mark his answer correct or explain why his answer is wrong.

- This is correct!

Worksheet 2-1: Q4

Let for $f(x, y) = 2x + 3y : S \rightarrow \mathbb{R}$ and $S = B[0, 1] = \{(x, y) : x^2 + y^2 \leq 1\} = \{(x, y) : \|(x, y)\|_2 \leq 1\}$. Answer the following questions.

- a) We call f a linear map if there is a matrix A such that $f(x, y) = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Find the matrix A to show that f is a linear map.

$$A = \begin{bmatrix} 2 & 3 \end{bmatrix}$$

- b) Note that because A is just a vector, $A\mathbf{x}$ is the same as the dot product. Use this and the Cauchy-Schwartz inequality to find $\arg \min_{x \in S} f(x)$ and $\arg \max_{x \in S} f(x)$

- The Cauchy-Schwartz inequality states that for any vectors \mathbf{u}, \mathbf{v} , $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|_2 \|\mathbf{v}\|_2$.
- Here, we can set $\mathbf{u} = A$ and $\mathbf{v} = \mathbf{x} = (x, y)$, so

$$|A \cdot \mathbf{x}| \leq \|A\|_2 \|\mathbf{x}\|_2.$$

- The left side is just our function, $|f(x, y)| = |A\mathbf{x}|$
- We know that $\|\mathbf{x}\|_2 \leq 1$ because $\mathbf{x} \in S$.
- We can also calculate $\|A\|_2 = \sqrt{2^2 + 3^2} = \sqrt{13}$.
- Putting this all together we have that

$$|f(x, y)| = |A\mathbf{x}| \leq \|A\|_2 \|\mathbf{x}\|_2 \leq \|A\|_2 = \sqrt{13}.$$

- Equality in Cauchy-Schwartz happens when the two vectors are parallel, so the maximum of $f(x, y)$ happens when \mathbf{x} is in the same direction as A . Thus,

$$\arg \max_{x \in S} f(x) = \frac{1}{\|A\|_2} A = \frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

- Similarly, the minimum happens when \mathbf{x} is in the opposite direction as A , so

$$\arg \min_{x \in S} f(x) = -\frac{1}{\|A\|_2} A = -\frac{1}{\sqrt{13}} \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

- Visual: desmos.com/3d/96keeizroq

- c) What do the points you found in part (b) represent?

These are the global min and global max, respectively.