Convex Functions: Part 2

Lecture 7-2 - CMSE 382

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This Lecture

Topics:

- Review Building Convex Functions
- Sublevel sets of convex functions
- CVXPY introduction

Announcements:

- Homework 3 due Friday!

Section 1

Review (Mostly)

Definition of convex functions

Definition

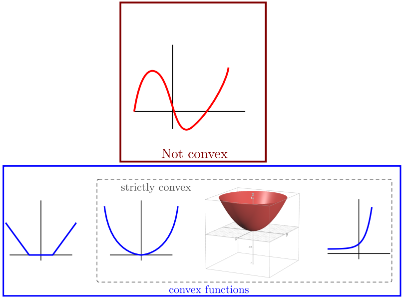
A function $f : C \to \mathbb{R}$ defined on a convex set $C \subset \mathbb{R}^n$ is

- **convex** if and only if

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$$
$$\forall \mathbf{x}, \mathbf{y} \in C, \lambda \in [0,1].$$

- **strictly convex** if and only if

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) < \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$$
$$\forall \mathbf{x} \neq \mathbf{y} \in C, \lambda \in (0,1).$$



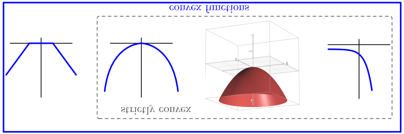
Definition of concave functions

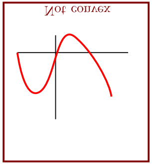
Definition

A function $f : C \to \mathbb{R}$ defined on a convex set $C \subset \mathbb{R}^n$ is **concave** if and only if

$$f(\lambda \mathbf{x} + (1-\lambda)\mathbf{y}) \geq \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y})$$
$$\forall \mathbf{x}, \mathbf{y} \in C, \lambda \in [0, 1].$$

Equivalently, f is concave if and only if $-f$ is convex.



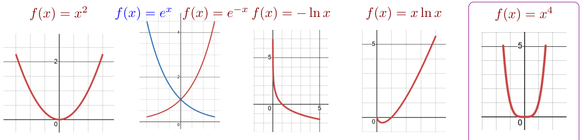


Second order characterization

Theorem

Let f be a twice continuously differentiable function over an open convex set $C \subseteq \mathbb{R}^n$.

- *f is convex over C if and only if $\nabla^2 f(\mathbf{x}) \succeq 0$ for any $\mathbf{x} \in C$.*
- *If $\nabla^2 f(\mathbf{x}) \succ 0$ for any $\mathbf{x} \in C$, then f is strictly convex over C.*



Strictly Convex
$\nabla f(x) = 2x$
$\nabla^2 f(x) = 2$

Strictly Convex

	e^x	e^{-x}
$\nabla f(x)$:	e^x	$-e^{-x}$
$\nabla^2 f(x)$:	e^x	e^{-x}

Strictly Convex
Over $x \in \mathbb{R}_{++}$
$\nabla f(x) = -1/x$
$\nabla^2 f(x) = 1/x^2$

Strictly Convex
Over $x \in \mathbb{R}_{++}$
$\nabla f(x) = \ln x + 1$
$\nabla^2 f(x) = 1/x$

Strictly Convex
$\nabla f = 4x^3$
$\nabla^2 f = 12x^2$
$\nabla^2 f(0) = 0$

Note: For 1D functions, this is just the second derivtive test (i.e., $f''(x) \geq 0$ for all x in the domain implies f is convex).

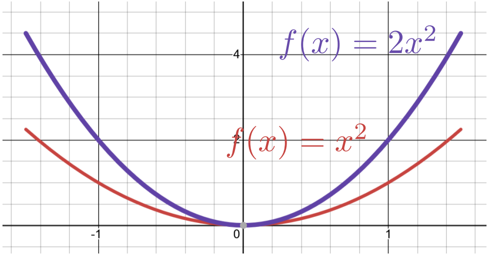
Some functions we know are convex

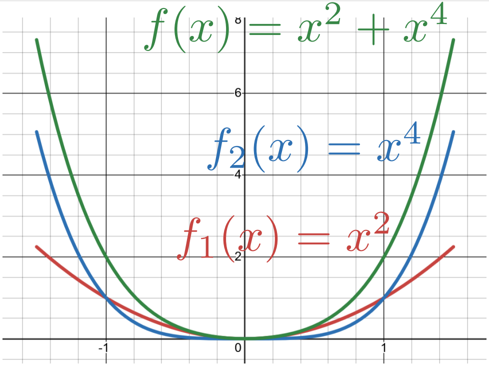
- Affine functions: $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x} + b$.
- Quadratic functions: $f(\mathbf{x}) = \mathbf{x}^\top A\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$ where $A \succeq 0$ is positive semidefinite.
- Norms: $f(\mathbf{x}) = \|\mathbf{x}\|$ for any norm on $\mathbb{R}^n$.
- Exponential function: $f(x) = \exp x$.

Theorem (Nonnegative scaling and sums)

Let $C \subseteq \mathbb{R}^n$ be convex.

- *If f is convex on C and $\alpha \geq 0$, then αf is convex on C.*
- *If $f_1, \dots, f_p$ are convex on C, then the sum $f_1 + \cdots + f_p$ is convex on C.*





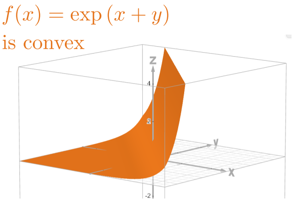
Affine change of variables

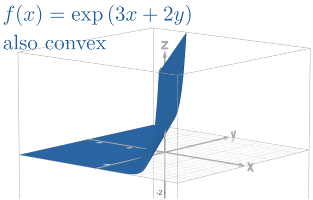
Theorem (Affine change of variables)

Let $f : C \to \mathbb{R}$ be convex on a convex set $C \subseteq \mathbb{R}^n$. Let $A \in \mathbb{R}^{n\times m}$ and $\mathbf{b} \in \mathbb{R}^n$. Define

$$g(\mathbf{y}) = f(A\mathbf{y} + \mathbf{b}),$$

on the convex set $D = \{\mathbf{y} \in \mathbb{R}^m : A\mathbf{y} + \mathbf{b} \in C\}$. Then g is convex on D.





Composition with monotone convex functions

Theorem (Convex outer, nondecreasing)

Let $f : C \to \mathbb{R}$ *be convex on a convex set* $C \subseteq \mathbb{R}^n$. *Let* $g : I \to \mathbb{R}$ *be a one-dimensional nondecreasing convex function defined on an interval* I, *and assume* $f(C) \subseteq I$. *Then the composition* $x \mapsto g(f(x))$ *is convex on* C.

Key idea: convexity is preserved when we apply a "nice" (convex, nondecreasing) 1D outer function to a convex inner function.

Section 2

Sublevel sets of convex functions

Convexity of sublevel sets of convex functions

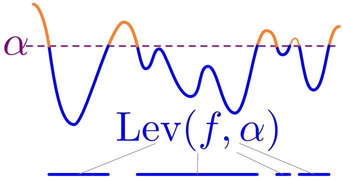
Definition

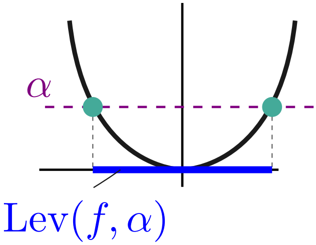
Let $f : S \to \mathbb{R}$ be a function defined over a set $S \subseteq \mathbb{R}^n$. Then **the sublevel set** of f with level α is given by

$$\text{SubLev}(f, \alpha) = \text{Lev}(f, \alpha) = \{x \in S : f(x) \leq \alpha\}$$

Theorem

Let $f : C \to \mathbb{R}$ be a convex function defined over a convex set $C \subseteq \mathbb{R}^n$. Then for any $\alpha \in \mathbb{R}$ the sublevel set SubLev(f, α) is convex.



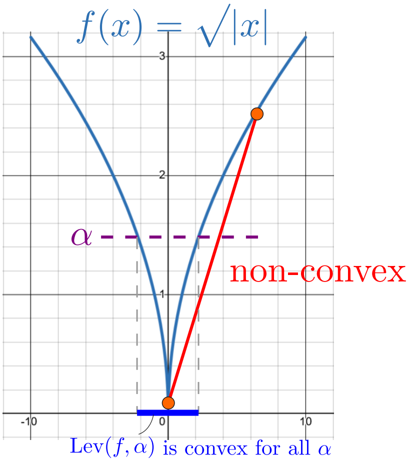


Quasi-convex functions

Definition

A function $f : C \to \mathbb{R}$ defined over the convex set $C \subseteq \mathbb{R}^n$ is called **quasi-convex** if for any $\alpha \in \mathbb{R}$ the set $\mathrm{Lev}(f, \alpha)$ is convex.

- A quasi-convex function may be non-convex.



Quasi-concave functions

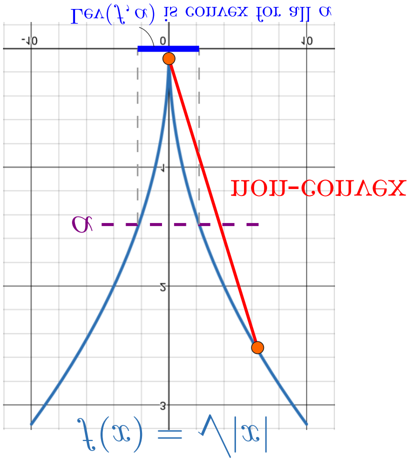
Definition

Define the **superlevel set** of f with level α as

$$\text{SupLev}(f, \alpha) = \{x \in S : f(x) \geq \alpha\}.$$

A function $f : C \to \mathbb{R}$ defined over the convex set $C \subseteq \mathbb{R}^n$ is called **quasi-concave** if for any $\alpha \in \mathbb{R}$ the set $\text{SupLev}(f, \alpha)$ is convex.

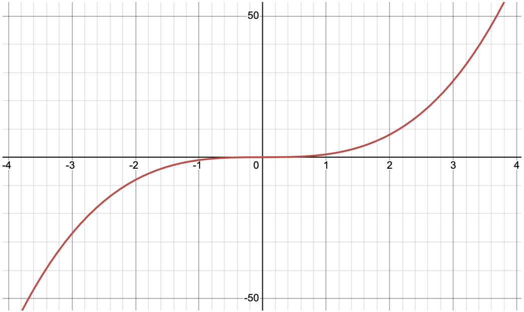
- A quasi-concave function may be non-concave.
- A function is quasi-concave if and only if its negative is quasi-convex.



Quasi-Linear

Definition

A function is **quasi-linear** if it is both quasi-convex and quasi-concave.



Example: $f(x) = x^3$ is quasi-linear but not linear, convex, or concave.

Groups - Round 3

Group 1
Lowell, Tianjuan, Lauryn, Atticus

Group 2
Alice, Aidan, Dev, Anthony

Group 3
Abigail, Michal, Breena, Andrew

Group 4
Kyle, Vinod, Dori, Joseph

Group 5
Yousif, Jamie, Jay, K.M Tausif

Group 6
Shanze, Saitej, Karen, Jack

Group 7
Arjun, Noah, Luis, Arya

Group 8
Morgan, Jonid, Sanskaar, Jake

Group 9
Quang Minh, Monirul Amin, Daniel, Ha

Group 10
Braedon, Dominic, Zheng, Lora

Group 11
Sai, Brandon, Purvi, Aaron

Group 12
Igor, Scott, Maye, Long