The KKT Conditions

Lecture 11-1 - CMSE 382

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Fri, April 3, 2026

This Lecture

Topics:

- Feasible descent direction
- Inequality and equality constrained problems
- Example: Equality constrained
- Example KKT not satisfied

Announcements:

- None

Outlook

$$\begin{aligned} &\text{minimize} && f(\mathbf{x}) \\ &\text{such that} && \mathbf{x} \in C \end{aligned}$$

Linearly Constrained

$$\begin{aligned} &\text{minimize} && f(\mathbf{x}) \\ &\text{such that} && \mathbf{a}_i^\top(\mathbf{x}) \leq b_i, \mathbf{c}_j^\top(\mathbf{x}) = \mathbf{d}_j, \\ & && i = 1, 2, \ldots, m. \\ & && j = 1, 2, \ldots, p. \end{aligned}$$

KKT Conditions

Inequality Constrained

$$\begin{aligned} &\text{minimize} && f(\mathbf{x}) \\ &\text{such that} && g_i(\mathbf{x}) \leq 0, \\ & && i = 1, 2, \ldots, m. \end{aligned}$$

KKT Conditions

Inequality & Equality Constrained

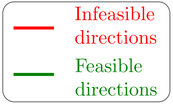
$$\begin{aligned} &\text{minimize} && f(\mathbf{x}) \\ &\text{such that} && g_i(\mathbf{x}) \leq 0, h_j(\mathbf{x}) = 0, \\ & && i = 1, 2, \ldots, m. \\ & && j = 1, 2, \ldots, p. \end{aligned}$$

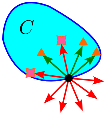
KKT Conditions

Section 1

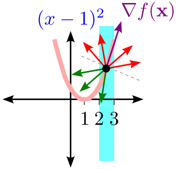
Inequality and equality constrained problems

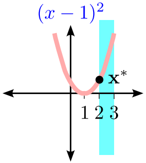
Idea: Feasible descent direction



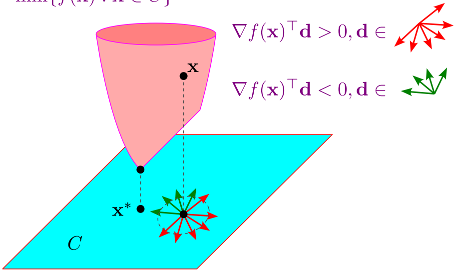


$\min\{(x-1)^2 : x \in [2,3]\}$





$\min\{f(\mathbf{x}) : \mathbf{x} \in C\}$



$f(▲) < f(●)$ $f(■) > f(●)$

$\mathbf{x} + t\mathbf{d} \notin C$ for all $t \in [0, \epsilon]$ and $\mathbf{d} \in$ 

No feasible descent directions

Feasible descent direction

Theorem (Feasible descent direction)

Consider the optimization problem

$$(P) \qquad \text{minimize} \quad f(\mathbf{x}) \qquad \text{such that} \quad \mathbf{x} \in C$$

where f is continuously differentiable function over the set $C \subseteq \mathbb{R}^n$.
Then a vector $\mathbf{d} \neq 0$ is called a **feasible descent direction at $\mathbf{x} \in C$** *if*

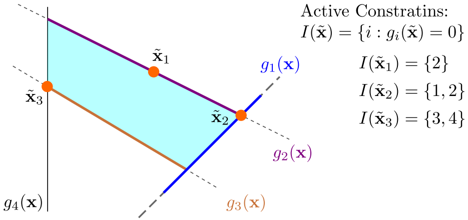
- $\nabla f(\mathbf{x})^\top \mathbf{d} < 0$, *and*
- *there exists $\varepsilon > 0$ such that $\mathbf{x} + t\mathbf{d} \in C$ for all $t \in [0, \varepsilon]$.*

Lemma

If $\mathbf{x}^*$ is a local optimal solution, then there are no feasible descent directions at $\mathbf{x}^*$.

Idea: there is no direction you can move that will both decrease the function's value and stay within the problem's constraints.

Recall: Active constraints



Definition (Recall: Active constraints)

Given a set of inequalities

$$g_i(\mathbf{x}) \le 0, \quad i = 1, 2, \ldots, m,$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ are functions, and a vector $\tilde{\mathbf{x}} \in \mathbb{R}^n$, the **active constraints at $\tilde{\mathbf{x}}$** are the constraints satisfied as equalities at $\tilde{\mathbf{x}}$. The set of active constraints is denoted by

$$I(\tilde{\mathbf{x}}) = \{i : g_i(\tilde{\mathbf{x}}) = 0\}.$$

Regular Points

Consider the minimization problem

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{such that} \quad & g_i(\mathbf{x}) \leq 0, i = 1, 2, \ldots m, \\ & h_j(\mathbf{x}) = 0, j = 1, 2, \ldots p. \end{aligned}$$

where $f, g_1, \ldots, g_m, h_1, h_2, \ldots, h_p$ are continuously differentiable functions over $\mathbb{R}^n$.

Definition

A feasible point $\mathbf{x}^*$ is called **regular** if the gradients of the active constraints among the inequality constraints and of the equality constraints

$$\{\nabla g_i(\mathbf{x}^*) \mid i \in I(\mathbf{x}^*)\} \\ \cup \{\nabla h_j(\mathbf{x}^*) \mid j = 1, \ldots, p\}$$

are linearly independent.

- Feasible points that are not regular are called **irregular** points.

Linear Independence

Definition

A collection of vectors

$$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$$

is **linearly independent** if the only solution to the equation

$$\alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \ldots + \alpha_k \mathbf{v}_k = \mathbf{0}$$

is $\alpha_1 = \alpha_2 = \ldots = \alpha_k = 0$.

Some methods for checking:

- Check using the definition directly.
- If there is one vector, this is linearly independent if it is nonzero.
- If there are two vectors, they are linearly independent if they are not scalar multiples of each other.
- Put the k vectors as columns in a matrix A.
  - If $k \leq n$ and $\mathrm{rank}(A) = k$, then the vectors are linearly independent.
  - If $k = n$, then the vectors are linearly independent if $\det(A) \neq 0$.
- If $k > n$, then the vectors are linearly dependent.

KKT Points

Consider the minimization problem

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{such that} \quad & g_i(\mathbf{x}) \leq 0, i = 1, 2, \ldots m, \\ & h_j(\mathbf{x}) = 0, j = 1, 2, \ldots p. \end{aligned}$$

where $f, g_1, \ldots, g_m, h_1, h_2, \ldots, h_p$ are continuously differentiable functions over $\mathbb{R}^n$.

Definition

A feasible point $\mathbf{x}^*$ is called a **KKT point** if there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ and $\mu_1, \mu_2, \ldots, \mu_p \in \mathbb{R}$ such that

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(\mathbf{x}^*) = \mathbf{0},$$

$$\lambda_i g_i(\mathbf{x}^*) = 0, i = 1, 2, \ldots, m.$$

KKT conditions for Inequality and equality constrained problems

Theorem (Inequality and equality constrained problems)

Let $\mathbf{x}^*$ *be a local minimum of the problem*

$$\begin{aligned} \textit{min} \quad & f(\mathbf{x}) \\ \textit{such that} \quad & g_i(\mathbf{x}) \leq 0, i = 1, 2, \ldots m, \\ & h_j(\mathbf{x}) = 0, j = 1, 2, \ldots p. \end{aligned}$$

where $f, g_1, \ldots, g_m, h_1, h_2, \ldots, h_p$ *are continuously differentiable functions over* $\mathbb{R}^n$. *Suppose that* $\mathbf{x}^*$ *is* **regular**, *then* $\mathbf{x}^*$ *is a* **KKT point**.

- A necessary condition for local optimality of a regular point is that it is a KKT point.
- Regularity is not required in the linearly constrained case.

Groups - Round 5

Group 1
Michal, Kyle, Daniel, Purvi

Group 2
Joseph, Jack, Scott, Breena

Group 3
Saitej, Dori, Noah, Tianjian

Group 4
Dev, Shanze, Lowell, Andrew

Group 5
Lora, Aidan, Arjun, Monirul Amin

Group 6
Anthony, Abigail, Atticus, Yousif

Group 7
Luis, Vinod, Morgan, Dominic

Group 8
Jay, Jonid, Alice, Aaron

Group 9
Arya, Jake, K M Tausif, Lauryn

Group 10
Maye, Ha, Zheng, Sai

Group 11
Jamie, Karen, Brandon, Quang Minh

Group 12
Long, Sanskaar, Braedon, Igor