Optimality Conditions for Linearly Constrained Problems

Lecture 10-3 - CMSE 382

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This Lecture

Topics:

- Orthogonal projection onto an affine space
- Orthogonal projection onto hyperplanes

Announcements:

- Quiz today!

Section 1

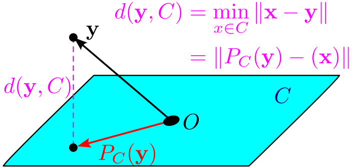
Orthogonal projection using KKT conditions

Recall: Orthogonal projection

Definition (Recall: Orthogonal projection operator)

Given a nonempty closed convex set C, the orthogonal projection operator $P_C : \mathbb{R}^n \to C$ is defined by

$$P_C(\mathbf{y}) = \arg\min \|\mathbf{x} - \mathbf{y}\|^2 : \mathbf{x} \in C.$$



- Returns the vector $\mathbf{x}$ in C that is closest to input vector $\mathbf{y}$.
- Is a convex optimization problem:

$$\begin{aligned} \min \quad & \|\mathbf{x} - \mathbf{y}\|^2 \\ \text{s.t.} \quad & \mathbf{x} \in C. \end{aligned}$$

Orthogonal Projection onto an Affine Space with KKT conditions

Let C be an affine space

$$\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}\},$$

where $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$.
Assume that the rows of A are linearly independent.
Given $\mathbf{y} \in \mathbb{R}^n$, find $P_C(\mathbf{y})$ which is the solution to the optimization problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x} - \mathbf{y}\|^2 \\ \text{s.t.} \quad & A\mathbf{x} = \mathbf{b}. \end{aligned}$$

- The Langrangian simplifies to

$$L(\mathbf{x}, \boldsymbol{\mu}) = \|\mathbf{x} - \mathbf{y}\|^2 + \boldsymbol{\mu}^\top (A\mathbf{x} - \mathbf{b}),$$

  for $\boldsymbol{\mu} \in \mathbb{R}^m$
- The KKT stationarity conditions are:

$$\nabla_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\mu}) = 2\mathbf{x} - 2\mathbf{y} + A^\top \boldsymbol{\mu} = \mathbf{0}$$

- Solving with $A\mathbf{x} = \mathbf{b}$ gives

$$P_C(\mathbf{y}) = \mathbf{y} - A^\top (AA^\top)^{-1}(A\mathbf{y} - \mathbf{b})$$

Orthogonal Projection onto a hyperplane with KKT conditions

Given a hyperplane

$$H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\top \mathbf{x} = b\}.$$

Given $\mathbf{y} \in \mathbb{R}^n$, find $P_H(\mathbf{y})$ which is the solution to the optimization problem

$$\begin{aligned} \min_{\mathbf{x}} \quad & \|\mathbf{x} - \mathbf{y}\|^2 \\ \text{s.t.} \quad & \mathbf{x} \in H \end{aligned}$$

- Special case of orthogonal projection onto an affine space with $A = \mathbf{a}^\top$ and $\mathbf{b} = b$.
- Replacing in the previous solution gives

$$\begin{aligned} P_H(\mathbf{y}) &= \mathbf{y} - \mathbf{a}(\mathbf{a}^\top \mathbf{a})^{-1}(\mathbf{a}^\top \mathbf{y} - b) \\ &= \mathbf{y} - \frac{\mathbf{a}^\top \mathbf{y} - b}{\|\mathbf{a}\|^2}\mathbf{a} \end{aligned}$$

Groups - Round 5

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