Optimality Conditions: Part 2

Lecture 2-2 - CMSE 382

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Topics covered

- Definite matrices and their eigenvalues
- Second order optimality conditions

Section 1

Definite matrices and their eigenvalues

Positive definite matrices

Definition

A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is

positive semidefinite	$A \succeq 0$	if $x^T A x \geq 0$ for every $x \in \mathbb{R}^n$
positive definite	$A \succ 0$	if $x^T A x > 0$ for every $x \neq 0$ in $\mathbb{R}^n$
negative semidefinite	$A \preceq 0$	if $x^T A x \leq 0$ for every $x \in \mathbb{R}^n$
negative definite	$A \prec 0$	if $x^T A x < 0$ for every $x \neq 0$ in $\mathbb{R}^n$
indefinite		if there is an $x \in \mathbb{R}^n$ such that $x^T A x > 0$ and a $y \in \mathbb{R}^n$ such that $y^T A y < 0$

Book always assume that a positive/negative definite/semidefinite matrix is symmetric.

Eigenvalue characterization theorem

Theorem

For a symmetric matrix $A \in \mathbb{R}^{n \times n}$, it is

positive semidefinite	*if and only if*	*all eigenvalues are nonnegative* (≥ 0)
positive definite	*if and only if*	*all eigenvalues are positive* (> 0)
negative semidefinite	*if and only if*	*all eigenvalues are nonpositive* (≤ 0)
negative definite	*if and only if*	*all eigenvalues are negative* (< 0)
indefinite	*if and only if*	*it has at least one positive and one negative eigenvalue*

Diagonal entries

Lemma

For a symmetric matrix $A \in \mathbb{R}^{n \times n}$,

if **positive semidefinite**	*then*	*all diagonal entries are nonnegative* (≥ 0)
if **positive definite**	*then*	*all diagonal entries are positive* (> 0)
if **negative semidefinite**	*then*	*all diagonal entries are nonpositive* (≤ 0)
if **negative definite**	*then*	*all diagonal entries are negative* (< 0)
if **indefinite**	*then*	*?????*

Warning: You cannot conclude positive/negative definiteness by checking the diagonal entries!

Section 2

Second order optimality condition

Recall: First order optimality condition

Let $f : U \to \mathbb{R}$ be a function defined on $U \subseteq \mathbb{R}^n$.
Suppose $x^* \in \text{int}(U)$ is a local optimum (minimum or maximum) point and that all partial derivatives of f exist at x^*.

Then $\nabla f(x^*) = 0$.

Second order optimality condition

Theorem (Second order optimality condition)

Let $f : U \to \mathbb{R}$ be twice continuously differentiable on an open set $U \subseteq \mathbb{R}^n$.
Let $\mathbf{x}^*$ be a stationary point of f (i.e., $\nabla f(\mathbf{x}^*) = \mathbf{0}$).
Then:

- **Necessary optimality conditions**:
  - *If $\mathbf{x}^*$ is a local minimum, then $\nabla^2 f(\mathbf{x}^*) \succeq 0$ (positive semi-definite)*
  - *If $\mathbf{x}^*$ is a local maximum, then $\nabla^2 f(\mathbf{x}^*) \preceq 0$ (negative semi-definite)*
- **Sufficient optimality conditions**:
  - *If $\nabla^2 f(\mathbf{x}^*) \succ 0$ (positive definite), then $\mathbf{x}^*$ is a strict local minimum*
  - *If $\nabla^2 f(\mathbf{x}^*) \prec 0$ (negative definite), then $\mathbf{x}^*$ is a strict local maximum*
- **Saddle point**:
  - *If $\nabla^2 f(\mathbf{x}^*)$ is an indefinite matrix, then $\mathbf{x}^*$ is a saddle point.*

Groups - Round 1

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Next time

- Check course webpage for videos and reading for next class
- Office hours posted on the course webpage