Name: **Present group members:**

Worksheet 1-1: Q1

For the L_p norm $\|x\|_p = \sqrt[p]{\sum_{i=1}^{n} |x_i|^p}$, we will check that the restriction $p \geq 1$ is necessary because for $0 \leq p < 1$ the function $\|\cdot\|_p$ is not a norm. Let's investigate this using the case $p = \frac{1}{2}$, where

$$\|x\|_{\frac{1}{2}} = \left(\sum_{i=1}^{n} |x_i|^{\frac{1}{2}} \right)^2 .$$

1. Write $\|x\|_{\frac{1}{2}}$ for each of the following vectors x:

   (a) e_1 *(b)* $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ *(c)* $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ *(d)* $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

   $(\sqrt{1})^2 = 1$ $(\sqrt{1} + \sqrt{1} + \sqrt{0} + \sqrt{1})^2 = 3^2 = 9$ $4^2 = 16$ $(\sqrt{1} + \sqrt{1})^2 = 2^2 = 4$

2. Let's see which of the norm properties (nonnegativity, positive homogeneity, triangle inequality) is violated for $p = \frac{1}{2}$.

   (a) Show that the nonnegativity property is satisfied for an arbitrary x.

   For any x_i entries, we take a square root of the absolute value, which is always positive. Summing positive numbers is positive, and then the square of that is positive. So the nonnegativity property is satisfied.

   (b) Show that the homogeneity property is satisfied for an arbitrary λ and x.

$$\|\lambda x\|_{0.5} = \left(\sum_{i=1}^{n} |\lambda x_i|^{\frac{1}{2}} \right)^2 = \left(\sum_{i=1}^{n} |\lambda|^{\frac{1}{2}} |x_i|^{\frac{1}{2}} \right)^2$$
$$= \left(|\lambda|^{\frac{1}{2}} \sum_{i=1}^{n} |x_i|^{\frac{1}{2}} \right)^2 = |\lambda| \left(\sum_{i=1}^{n} |x_i|^{\frac{1}{2}} \right)^2 = |\lambda| \|x\|_{0.5}.$$

   So the homogeneity property is satisfied.

   (c) Show the triangle inequality is violated. Specifically, consider the two standard basis e_1 and e_2 and check whether $\|e_1 + e_2\|_{0.5} \leq \|e_1\|_{0.5} + \|e_2\|_{0.5}$

   The left side is $\|(1,1)\|_{0.5} = 4$ from above, while the right side is $\|e_1\|_{0.5} + \|e_2\|_{0.5} = 1 + 1 = 2$. Since $4 \not\leq 2$, the triangle inequality is violated.

Worksheet 1-1: Q2

Let's look a visual representation of planar vectors whose L_p norm is equal to 1.

1. Start with L_1 norm. Four vectors that satisfy $\|x\| = 1$ include the vectors $(-1, 0)$, $(0, -1)$, $(1, 0)$, and $(0, 1)$. The points of the arrowhead for each of these vectors are labeled in part (a) of the figure below. For each of the following questions, find the missing x or y entry of the given vector that will make its L_1 norm equal to 1.

   (a) $(0.1, y)$ (b) $(x, 0.5)$ (c) $(0.75, y)$

   This is 0.9, 0.5, *and* 0.25 *respectively.*

2. Draw a geometric shape that shows all the possible end points of vectors that satisfy $\|x\|_1 = 1$.

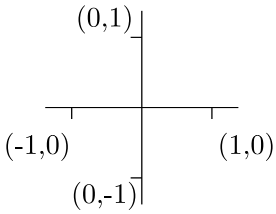
   The trick is $\|(x, y)\|_1 = |x| + |y| = 1$, *so this should be a diamond shape below.*

3. Now, let's work the L_2 norm. Write the L_2 norm equation for two arbitrary, planar vectors x and y and set it to 1. Does the resulting equation remind you of the equation of a geometric shape? Draw the resulting geometric shape in part (b) of the figure below.

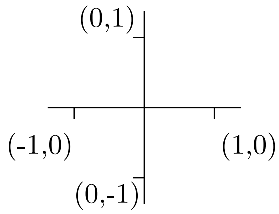
   This is a circle

4. Finally, write the equation that describes $\|x\|_\infty = 1$. What is the requirement on the x y components of a planar vector to satisfy this equation? Draw the geometric shape that represents the arrowheads of all the vectors for which $\|x\|_\infty = 1$.

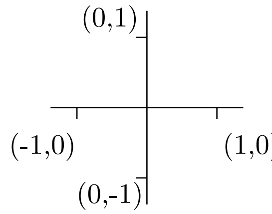
   $\|x\|_\infty = \max\{|x_1|, |x_2|\}$. *This ends up with a square with corners at* $(\pm 1, \pm 1)$.



(a) L_1



(b) L_2



(c) L_∞

Worksheet 1-1: Q3

Recall the Cauchy-Schwarz inequality: For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we have $|\mathbf{x}^\top \mathbf{y}| \leq \|\mathbf{x}\|_2 \cdot \|\mathbf{y}\|_2$. Let's see when we get the equality.

1. The left hand side represents the dot product. Recall $x \cdot y = \|x\|\|y\| \cos\theta$, where θ is the angle between the two vectors. Based on this, when will the two sides of the inequality be equal? *We have $\|x\| \cdot \|y\| \cos\theta \leq \|x\|\|y\|$, so you just need $\cos\theta = 1$. This happens when $\theta = 0, \pi$*

2. Sketch two vectors in $\mathbb{R}^2$ for which the equality holds. *These just have to either be pointing the same direction of opposite directions.*