Name: **Present group members:**

Worksheet 1-2: Q1 Circle all the properties of each given set.

1. $S_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$

   Open Closed Bounded Compact

   Interior of disk: Open and bounded

2. $S_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

   Open Closed Bounded Compact

   Closed disk: Closed and bounded, so also compact

3. $S_3 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

   Open Closed Bounded Compact

   Circle: Closed and bounded, so also compact

4. $S_4 = [0, 1) \subset \mathbb{R}$

   Open Closed Bounded Compact

   Half-open interval: Bounded, but neither open nor closed, so not compact

5. $S_5 = (0, 1) \cup (2, 3) \subset \mathbb{R}$

   Open Closed Bounded Compact

   Union of open intervals: Open and bounded

6. $S_6 = [0, 1] \cup [2, 3] \subset \mathbb{R}$

   Open Closed Bounded Compact

   Union of closed intervals: Closed and bounded, so also compact

7. $S_7 = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 1\}$

   Open Closed Bounded Compact

   First quadrant above $y = 1$: Closed, but unbounded, so not compact

Worksheet 1-2: Q2

1. Find the gradient for the scalar-valued function $\mathbf{f}(x, y, z) = x^4 + 3yz$ at $(1, 2, 3)$.

   - Compute partial derivatives:

$$\nabla f(x,y,z) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 4x^3 \\ 3z \\ 3y \end{bmatrix}$$

   - Evaluate at given point:

$$\nabla f(1,2,3) = \begin{bmatrix} 4(1)^3 \\ 3(3) \\ 3(2) \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 6 \end{bmatrix}$$

2. Find the directional derivative of the function at $(1, 2, 3)$ in the direction of a unit vector parallel to $\mathbf{u} = (1, -2, 2)$.

   - Find the unit vector in the direction of $(1, -2, 2)$ by dividing by the norm.

$$\|(1,-2,2)\| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3 \implies \hat{\mathbf{u}} = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

   - Compute the directional derivative using the gradient and the unit vector:

$$\nabla f(1,2,3) \cdot \hat{\mathbf{u}} = \begin{bmatrix} 4 & 9 & 6 \end{bmatrix} \cdot \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right) = \frac{4}{3} - \frac{18}{3} + \frac{12}{3} = \frac{4 - 18 + 12}{3} = \frac{-2}{3}$$

3. Find the Hessian for $\mathbf{f}$ at $\mathbf{x} = (1, 2, 3)$

   - First, find the Hessian:

$$\nabla^2 f(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{pmatrix} = \begin{pmatrix} 12x^2 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$

   - Then evaluate at the given point:

$$\nabla^2 f(1,2,3) = \begin{pmatrix} 12(1)^2 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{pmatrix}$$

Worksheet 1-2: Q3 Consider the function $f(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$ and the point $\mathbf{x} = (1, 1)$.

1. Compute $\nabla f(\mathbf{x})$ at $\mathbf{x} = (1, 1)$.
   $\frac{\partial f}{\partial x_1} = 2x_1 + 2x_2 = 4$ *and* $\frac{\partial f}{\partial x_2} = 2x_1 + 2x_2 = 4,$ *so* $\nabla f(1, 1) = (4, 4)^T$

2. Write the linear approximation $L(\mathbf{y})$ to f at $\mathbf{x} = (1, 1)$:

$$L(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^T (\mathbf{y} - \mathbf{x})$$

   First, $f(1, 1) = 1 + 2 + 1 = 4.$ *Then* $L(y_1, y_2) = 4 + 4(y_1 - 1) + 4(y_2 - 1) = 4y_1 + 4y_2 - 4$

3. Use the linear approximation $L(\mathbf{y})$ you just calculated to estimate $f(1.1, 0.9)$.
   $L(1.1, 0.9) = 4(1.1) + 4(0.9) - 4 = 4.4 + 3.6 - 4 = 4$

4. Compute the actual value $f(1.1, 0.9)$.
   $f(1.1, 0.9) = (1.1)^2 + 2(1.1)(0.9) + (0.9)^2 = 1.21 + 1.98 + 0.81 = 4.00$

5. What is the error in the linear approximation? (i.e., $|f(1.1, 0.9) - L(1.1, 0.9)|$)
   Error $= |4.00 - 4| = 0.00$

6. Compute the Hessian matrix $\nabla^2 f(\mathbf{x})$ at $\mathbf{x} = (1, 1)$.

   $\frac{\partial^2 f}{\partial x_1^2} = 2,\ \frac{\partial^2 f}{\partial x_1 \partial x_2} = 2,\ \frac{\partial^2 f}{\partial x_2^2} = 2,$ *so* $\nabla^2 f(1, 1) = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

7. For $\mathbf{y} = (1.1, 0.9)$, compute the (approximate) quadratic error term:

$$E_2(\mathbf{y}) \approx \frac{1}{2}(\mathbf{y} - \mathbf{x})^T \nabla^2 f(\mathbf{x})(\mathbf{y} - \mathbf{x})$$

   - $(\mathbf{y} - \mathbf{x}) = (0.1, -0.1)^T$.
   - Then $E_2 = \frac{1}{2}(0.1, -0.1) \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 0.1 \\ -0.1 \end{pmatrix} = \frac{1}{2}(0.1, -0.1) \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0$

8. For $\mathbf{y} = (1.1, 1.1)$, compute the (approximate) quadratic error term:

$$E_2(\mathbf{y}) \approx \frac{1}{2}(\mathbf{y} - \mathbf{x})^T \nabla^2 f(\mathbf{x})(\mathbf{y} - \mathbf{x})$$

   - $(\mathbf{y} - \mathbf{x}) = (0.1, 0.1)^T$.
   - $E_2(1.1, 1.1) = \frac{1}{2}(0.1, 0.1) \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 0.1 \\ 0.1 \end{pmatrix} = \frac{1}{2}(0.1, 0.1) \begin{pmatrix} 0.4 \\ 0.4 \end{pmatrix} = \frac{1}{2}(0.04 + 0.04) = 0.04$

9. What does an error $E_2(\mathbf{y}) = 0$ tell us? What does a non-zero error term tell us? What does the error depend on?

   Hint: Take a look at this Desmos to get some intuition: https://www.desmos.com/3d/esp2pdudke

   The Hessian has a null space! The direction $(1, -1)$ is an eigenvector with eigenvalue 0, so the curvature vanishes in that direction. Moving along $(0.1, -0.1)$ means we're moving in a direction where the function has zero curvature, so the quadratic approximation is exact. This shows that the error depends on both the distance and *the direction you move.*