Duality

Lecture 12-1 - CMSE 382

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Fri, April 10, 2026

This Lecture

Topics:

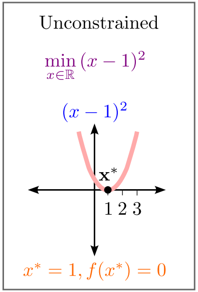
- Motivation for Duality
- Definition and weak duality

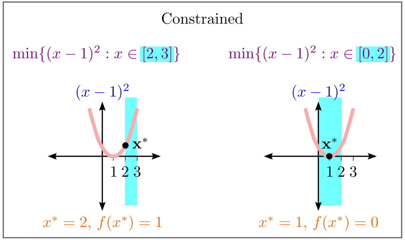
Announcements:

- Homework 5 is due today, Friday April 10, at 11:59pm.

Section 1

Motivation





- The optimal value (val) of unconstrained problem is lower bound for the constrained one.
  - val(*unconstrained*) $\leq$ val(*constrained*)
- Interested in finding lower bounds for constrained optimization by solving unconstrained problems.

Obtaining lower bounds

Approach#1: Drop constraints

$$\begin{aligned} \min \quad & f(\mathbf{x}) = x^2 + y^2 + 2x \\ \text{such that} \quad & x + y = 0. \end{aligned}$$

$$\min \quad f(\mathbf{x}) = x^2 + y^2 + 2x$$

- val(*unconstrained*) $\leq$ val(*constrained*)
- May not be the best lower bound.
- We want the largest lower bound.

Desmos example

Obtaining lower bounds

Approach#2: Optimize for best lower bounds

$$
\begin{aligned}
(\text{P}) \quad \min \quad & f(\mathbf{x}) + \mu h(\mathbf{x}) \\
& = x^2 + y^2 + 2x + \mu(x+y) \\
\text{s.t.} \quad & h(\mathbf{x}) = x + y = 0.
\end{aligned}
$$

$$
\begin{aligned}
(\text{P}_\mu) \quad \min \quad & f(\mathbf{x}) + \mu h(\mathbf{x}) \\
& = x^2 + y^2 + 2x + \mu(x+y)
\end{aligned}
$$

- $\text{val}(\text{P}_\mu) \leq \text{val}(\text{P})$ for all $\mu \in \mathbb{R}$.
- Best lower bound is the solution to

$$
(\text{D}) \qquad \max_{\mu} \{\text{val}(P_\mu)\}.
$$

Desmos example

Primal and Dual Problems

Primal Problem

$$
\text{(P)} \quad \min \quad f(\mathbf{x}) \quad \text{such that} \quad h(\mathbf{x}) = 0.
$$

Dual Problem

$$
\text{(D)} \quad \max \quad \text{val}(P_\mu)
$$

Section 2

Duality Definition

Dual objective function

Consider the general model referred to as the primal model

$$
\begin{aligned}
f^* = \min \quad & f(\mathbf{x}) \\
\text{such that} \quad & g_i(\mathbf{x}) \leq 0, i = 1, 2, \dots m, \\
& h_j(\mathbf{x}) = 0, j = 1, 2, \dots p, \\
& \mathbf{x} \in X, \text{ where } X \subseteq \mathbb{R}^n,
\end{aligned}
$$

and f, g_i, h_j are functions defined on X.

The Lagrangian of the problem is

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \mu_j h_j(\mathbf{x}),$$

The dual objective function $q : \mathbb{R}_+^m \times \mathbb{R}^p \to \mathbb{R} \cup \{-\infty\}$ is

$$q(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \min_{x \in X} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}),$$

Dual problem

Definition (Dual Problem)

The dual problem is given by

$$\begin{aligned} q^* = \max_{\boldsymbol{\lambda},\boldsymbol{\mu}} & \quad q(\boldsymbol{\lambda},\boldsymbol{\mu}) \\ \text{such that} & \quad (\boldsymbol{\lambda},\boldsymbol{\mu}) \in \text{dom}(q), \end{aligned}$$

where the domain of the dual objective function is

$$\text{dom}(q) = \{(\boldsymbol{\lambda},\boldsymbol{\mu}) \in \mathbb{R}^m_+ \times \mathbb{R}^p : q(\boldsymbol{\lambda},\boldsymbol{\mu}) > -\infty\}.$$

Convexity of the dual problem

Theorem (Convexity of the dual problem)

Let the dual problem be given by

$$q^* = \max_{\boldsymbol{\lambda},\boldsymbol{\mu}} \quad q(\boldsymbol{\lambda},\boldsymbol{\mu})$$
$$\textit{such that} \quad (\boldsymbol{\lambda},\boldsymbol{\mu}) \in dom(q),$$

where $f, g_1, \dots, g_m, h_1, \dots, h_p$ *are functions defined on* $X \subseteq \mathbb{R}^n$*, and* $q(\boldsymbol{\lambda},\boldsymbol{\mu}) = \min_{\mathbf{x}\in X} L(\mathbf{x},\boldsymbol{\lambda},\boldsymbol{\mu})$*. Then*

- (a) *$dom(q)$ is a convex set.*
- (b) *q is a concave function over $dom(q)$.*

Maximizing a concave function over a convex set defines a convex problem.

Weak duality theorem

Primal Problem

$$\begin{aligned} f^* = \min \quad & f(\mathbf{x}) \\ \text{such that} \quad & g_i(\mathbf{x}) \leq 0, i = 1, 2, \ldots m, \\ & h_j(\mathbf{x}) = 0, j = 1, 2, \ldots p, \\ & \mathbf{x} \in X, \ \text{where } X \subseteq \mathbb{R}^n, \end{aligned}$$

and f, g_i, h_j are functions defined on X.

Dual Problem

$$\begin{aligned} q^* = \max \quad & q(\boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \text{such that} \quad & (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \text{dom}(q), \end{aligned}$$

where $\text{dom}(q) = \{(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{R}_+^m \times \mathbb{R}^p : q(\boldsymbol{\lambda}, \boldsymbol{\mu}) > -\infty\}$, and $q(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \min_{\mathbf{x} \in X} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$.

Theorem (Weak duality theorem)

Consider the primal problem and its dual. Then $q^* \leq f^*$, where q^*, f^* are the optimal dual and primal values, respectively.

Groups - Round 5

Group 1
Michal, Kyle, Daniel, Purvi

Group 2
Joseph, Jack, Scott, Breena

Group 3
Saitej, Dori, Noah, Tianjian

Group 4
Dev, Shanze, Lowell, Andrew

Group 5
Lora, Aidan, Arjun, Monirul Amin

Group 6
Anthony, Abigail, Atticus, Yousif

Group 7
Luis, Vinod, Morgan, Dominic

Group 8
Jay, Jonid, Alice, Aaron

Group 9
Arya, Jake, K M Tausif, Lauryn

Group 10
Maye, Ha, Zheng, Sai

Group 11
Jamie, Karen, Brandon, Quang Minh

Group 12
Long, Sanskaar, Braedon, Igor