Name: **Present group members:**

Worksheet 12-1: Q1

Consider the problem

$$\begin{aligned} f^* = \min \quad & x_1^2 - x_2 \\ \text{s.t.} \quad & x_1 - x_2^2 = 0, \end{aligned}$$

1. Find the optimal solution by replacing x_1 with x_2^2 and making the objective a single variable function. What is the optimal value f^*? (Note f^* is the minimimal value of the objection function $f(x_1, x_2) = x_1^2 - x_2$, not the point x_1, x_2 that achieves the minimum.)

2. Now solve the problem using duality:

   (a) Write down the Lagrangian.

   (b) What is $\nabla_{x_1,x_2} L(x_1, x_2, \mu)$? When is $\nabla_{x_1,x_2} L(x_1, x_2, \mu) = 0$?

   (c) Write a formula for $q(\mu) = \min_{x_1,x_2} L(x_1, x_2, \mu)$ in terms of μ.

(d) What value of μ maximizes $q(\mu)$?

(e) What is the optimal dual value $q^* = \max_\mu q(\mu)$?

3. We know from the weak duality theorem that $q^* \leq f^*$. Is the bound tight for this problem? That is, do we have $q^* = f^*$ in this problem?

Worksheet 12-1: Q2

Consider the problem

$$\begin{aligned} \min \quad & x_1^2 - x_2 \\ \text{s.t.} \quad & x_2^2 \leq 0, \end{aligned}$$

1. Find the optimal solution by reducing the problem to a single variable unconstrained optimization problem without using duality.

2. Write down the Lagrangian.

3. Obtain the dual objective function $q(\lambda) = \min_{x_1,x_2} L(x_1, x_2, \lambda)$. Note that you will need to consider the two cases: $\lambda = 0$ and $\lambda > 0$. This means your objective function will be a piece-wise function.

4. What is $q^* = \max_{\lambda \geq 0} q(\lambda)$?

5. Is the bound from the weak duality theorem tight for this problem? That is, do we have $q^* = f^*$ in this problem?

Worksheet 12-1: Q3

$$(\text{P}) \quad \min_{(x,y)} x^2 + y^2$$
$$\text{s.t. } 1 - x^4 \leq 0$$

1. Using the desmos plot, what do you think the optimal solution f^* is?

2. Now solve the problem using duality:

   (a) Write down the Lagrangian.

   (b) In a different Desmos plot from above, plot the Lagrangian as a function of x and y using a slider bar for different values of $\lambda \geq 0$. What do you observe? What does this suggest about the dual function $q(\lambda) = \min_{x,y} L(x, y, \lambda)$? What is $q^* = \max_\lambda q(\lambda)$?
   For an additional challenge, can you justify your observation without using Desmos?

3. Based on what you found above, is the bound from the weak duality theorem tight for this problem? That is, do we have $q^* = f^*$ in this problem?