Optimality Conditions: Part 3

Lecture 2-3 - CMSE 382

Prof. Elizabeth Munch

Michigan State University
::
Dept of Computational Mathematics, Science & Engineering

Mon, Jan 26, 2026

Topics covered

- Coercive functions
- Quadratic functions
- Convex functions
- Global optimality

Section 1

Global optimality

Coerciveness

Definition

A continuous function defined over $\mathbb{R}^n$ is **coercive** if

$$\lim_{\|\mathbf{x}\| \to +\infty} f(\mathbf{x}) = \infty.$$

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous and coercive function, and let $S \subseteq \mathbb{R}^n$ be a closed set. Then f has a **global minimum** *over S.*

Example functions desmos.com/3d/ajb6tdoryd

Quadratic functions

Definition

A **quadratic function** over $\mathbb{R}^n$ is a function of the form

$$f(\mathbf{x}) = \mathbf{x}^\top \mathbf{A}\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + c,$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric, $\mathbf{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$.

For all $\mathbf{x} \in \mathbb{R}^n$,

$$\nabla f(\mathbf{x}) = 2\mathbf{A}\mathbf{x} + 2\mathbf{b}$$
$$\nabla^2 f(\mathbf{x}) = 2\mathbf{A}$$

Let $\mathbf{A} = \begin{bmatrix} a_1 & a_2 \\ a_2 & a_4 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, and $c \in \mathbb{R}$.

$$\begin{aligned} f(x_1, x_2) &= [x_1 x_2] \begin{bmatrix} a_1 & a_2 \\ a_2 & a_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2 \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + c \\ &= a_1 x_1^2 + 2a_2 x_1 x_2 + a_4 x_2^2 + 2b_1 x_1 + 2b_2 x_2 + c \end{aligned}$$

Properties of quadratic functions

Properties of quadratic functions

Let $f(\mathbf{x}) = \mathbf{x}^\top \mathbf{A}\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + c$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric, $\mathbf{b} \in \mathbb{R}^n$, and $c \in \mathbb{R}$. Then:

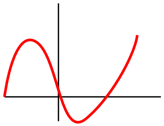
- a. $\mathbf{x}$ is a stationary point of f if and only if $\mathbf{A}\mathbf{x} = -\mathbf{b}$.
- b. If $\mathbf{A} \succeq 0$, then $\mathbf{x}$ is a global minimum point of f if and only if $\mathbf{A}\mathbf{x} = -\mathbf{b}$.
- c. If $\mathbf{A} \succ 0$ then $\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b}$ is a strict global minimum point of f.
- d. f is coercive if and only if $\mathbf{A} \succ 0$.
- e. $f(\mathbf{x}) \geq 0$ if and only if $\begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^\top & c \end{bmatrix} \succeq 0$

Convex functions

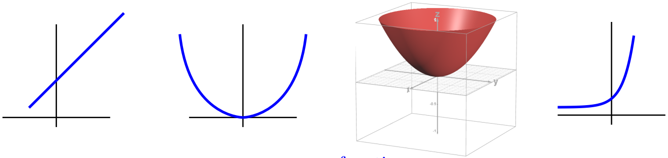
Convex Functions

Let f be a twice continuously differentiable function defined over $\mathbb{R}^n$. Suppose that $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ for any $\mathbf{x} \in \mathbb{R}^n$.

- If $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ for any $x \in \mathbb{R}^n$, we call the function **convex** (generalization of 'cup-shaped' real functions on $\mathbb{R}$.)



Not convex

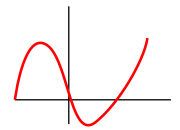


convex functions

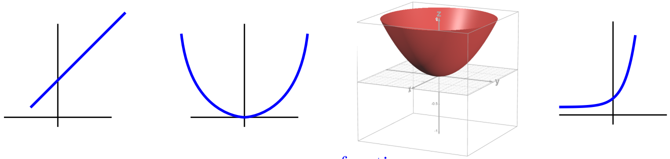
Theorem

Let f be a twice continuously differentiable function defined over $\mathbb{R}^n$. Suppose that f is convex over its domain, i.e., $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ for any $\mathbf{x} \in \mathbb{R}^n$, and let $\mathbf{x}^*$ be a stationary point of f. Then:

- $\mathbf{x}^*$ *is a* **global minimum** *of f.*
- **A local optimum of f is a global optimum.**



Not convex



convex functions

Groups - Round 2

Group 1
Abigail, Shanze, Jack, Quang Minh,

Group 2
Igor, Atticus, K M Tausif, Long,

Group 3
Yousif, Zheng, Jake, Purvi,

Group 4
Maye, Alice, Arjun, Kyle,

Group 5
Monirul Amin, Jay, Brandon, Luis,

Group 6
Scott, Ha, Lora, Tianjian,

Group 7
Braedon, Sai, Joseph, Noah,

Group 8
Michal, Aidan, Jonid, Dev,

Group 9
Vinod, Saitej, Anthony, Breena,

Group 10
Karen, Dori, Lowell, Aaron,

Group 11
Jamie, Sanskaar, Dominic, Lauryn,

Group 12
Andrew, Arya, Daniel, Morgan,

Next time

- Check course webpage for videos and reading for next class
- Office hours posted on the course webpage
- Bring computer for jupyter notebook work next class!
- Homework posted, due Friday Jan 30 at 11:59pm
- Quiz 2 on Weds 2/4