Name: **Present group members:**

Worksheet 10-1: Q1

Let $m, n = 2$, and the matrices

$$\mathbf{A} = \begin{bmatrix} 6 & 3 \\ 4 & 0 \end{bmatrix}, \qquad \mathbf{c} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

1. The first possibility from Farkas' lemma is that there exists $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{A}\mathbf{x} \leq \mathbf{0}$ and $\mathbf{c}^T\mathbf{x} > 0$.

   (a) Write down the system of inequalities implied by $\mathbf{A}\mathbf{x} \leq \mathbf{0}$. Use Desmos to sketch the region defined by these inequalities.

   - *The assumption $\mathbf{A}\mathbf{x} \leq \mathbf{0}$ means*

   $$\begin{aligned} 6x_1 + 3x_2 &\leq 0, \\ 4x_1 &\leq 0. \end{aligned}$$

   (b) Write down the inequality implied by $\mathbf{c}^T\mathbf{x} > 0$. Draw this restriction on the same Desmos plot.

   - *The assumption $\mathbf{c}^T\mathbf{x} > 0$ means $3x_1 + x_2 > 0$.*

   (c) Does there exist $\mathbf{x} \in \mathbb{R}^2$ such that $\mathbf{A}\mathbf{x} \leq \mathbf{0}$ and $\mathbf{c}^T\mathbf{x} > 0$? Justify your answer using the Desmos plot.

   - *No. See my Desmos plot here.*
   - *The region defined by $\mathbf{A}\mathbf{x} \leq \mathbf{0}$ is the interseciton of the blue and red regions in the plot.*
   - *The region defined by $\mathbf{c}^T\mathbf{x} > 0$ is the green region in the plot.*
   - *Since there is no overlap between the blue/red region and the green region, there is no $\mathbf{x}$ that satisfies both conditions.*

2. The second possibility from Farkas' lemma is that there exists $\mathbf{y} \in \mathbb{R}^2$ such that $\mathbf{A}^\top\mathbf{y} = \mathbf{c}$ and $\mathbf{y} \geq 0$.

   (a) Write down the system of equations implied by $\mathbf{A}^\top\mathbf{y} = \mathbf{c}$. Plot the solution these equations in a new Desmos plot.
   The assumption $\mathbf{A}^\top\mathbf{y} = \mathbf{c}$ means

   $$\begin{aligned} 6y_1 + 4y_2 &= 3, \\ 3y_1 + 0y_2 &= 1. \end{aligned}$$

   (b) Is there a solution $\mathbf{y} \geq 0$ to the equations above? Use your Desmos plot to justify.
   The lines intersect at $\mathbf{y} = (1/3, 1/4)$, see my Desmos plot here. Since both entries are positive, $\mathbf{y} \geq 0$. So there is a solution $\mathbf{y} \geq 0$ to the equations above.

(c) Based on the last page, which of the two possibilities from Farkas' lemma holds for the matrices $\mathbf{A}$ and $\mathbf{c}$ above? Justify your answer.

- *The second possibility from Farkas' lemma holds for the matrices* $\mathbf{A}$ *and* $\mathbf{c}$ *above.*
- *We showed in part (a) that there is no* $\mathbf{x}$ *such that* $\mathbf{A}\mathbf{x} \le \mathbf{0}$ *and* $\mathbf{c}^T\mathbf{x} > 0$. *So the first possibility does not hold.*
- *We showed in part (b) that there is a* $\mathbf{y}$ *such that* $\mathbf{A}^\top\mathbf{y} = \mathbf{c}$ *and* $\mathbf{y} \ge 0$. *So the second possibility does hold.*

(d) Repeat the previous part for the matrices

$$\mathbf{A} = \begin{bmatrix} 6 & 3 \\ 4 & 0 \end{bmatrix}, \qquad \mathbf{d} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

Which of the two possibilities from Farkas' lemma holds for the matrices $\mathbf{A}$ and $\mathbf{d}$ above? Justify your answer.

- *For the first option of Farkas' lemma:*
    - *The inequalities from* $\mathbf{A}\mathbf{x} \le 0$ *are the same:*

      $$\begin{aligned} 6x_1 + 3x_2 &\le \mathbf{0}, \\ 4x_1 &\le 0. \end{aligned}$$

      They are the red and green regions of this Desmos plot.
    - *The inequality from* $\mathbf{d}^T\mathbf{x} > 0$ *is* $x_1 - x_2 > 0$, *which is the green region in the plot above.*
    - *There are lots of options for* $\mathbf{x}$ *that satisfy both conditions. For example,* $\mathbf{x} = (-2, -4)$ *satisfies both conditions. So the first option of Farkas' lemma holds for the matrices* $\mathbf{A}$ *and* $\mathbf{d}$ *above.*
- *The first option of Farkas' lemma holds so we know the second doesn't. But just for the sake of practice, we can check.*
    - $\mathbf{A}^\top\mathbf{y} = \mathbf{d}$ *means*

      $$\begin{aligned} 6y_1 + 4y_2 &= 1, \\ 3y_1 + 0y_2 &= -1. \end{aligned}$$

    - *See this Desmos plot.*
    - *The lines intersect at* $(-1/3, 3/4)$, *which is not non-negative. So there is no* $\mathbf{y} \ge 0$ *such that* $\mathbf{A}^\top\mathbf{y} = \mathbf{d}$.

Worksheet 10-1: Q2 Find the stationary point(s) for

$$\min \frac{1}{2}\left(x_1^2+x_2^2+x_3^2\right) \text{ s.t. } x_1+x_2+x_3=3$$

by following the steps below.

(a) Determine f, A, $\mathbf{b}$, C, and $\mathbf{d}$ to write the problem in standard form,

$$\min_{\mathbf{x}} f(\mathbf{x}) \qquad \text{s.t.} \quad A\mathbf{x} \leq \mathbf{b},\ C\mathbf{x}=\mathbf{d}.$$

- $f(\mathbf{x})=\frac{1}{2}\left(x_1^2+x_2^2+x_3^2\right)$
- $A=\mathbf{0}$, $\mathbf{b}=\mathbf{0}$ *(since there are no inequality constraints)*
- $C=\begin{bmatrix}1 & 1 & 1\end{bmatrix}$, $\mathbf{d}=3$ *(since the equality constraint is $x_1+x_2+x_3=3$)*

(b) Write down the Lagrangian function,

- *The Lagrangian function is*

$$L(\mathbf{x},\mu)=f(\mathbf{x})+\boldsymbol{\mu}(A\mathbf{x}-\mathbf{b})+\boldsymbol{\mu}^\top(C\mathbf{x}-\mathbf{d}).$$

- *Since $A=\mathbf{0}$ and $\mathbf{b}=\mathbf{0}$, the Lagrangian simplifies to*

$$L(\mathbf{x},\mu)=f(\mathbf{x})+\boldsymbol{\mu}^\top(C\mathbf{x}-\mathbf{d})=\frac{1}{2}\left(x_1^2+x_2^2+x_3^2\right)+\mu(x_1+x_2+x_3-3).$$

(c) Write down the KKT condition (also called the stationarity condition).

- *The KKT condition is*

$$\nabla_{\mathbf{x}}L(\mathbf{x},\mu)=\nabla f(\mathbf{x})+A^\top\boldsymbol{\lambda}+C^\top\boldsymbol{\mu}=\mathbf{0}.$$

- *Since $A=\mathbf{0}$, the stationarity condition simplifies to*

$$\nabla_{\mathbf{x}}L(\mathbf{x},\mu)=\nabla f(\mathbf{x})+C^\top\boldsymbol{\mu}=\mathbf{0}.$$

- *We can compute $\nabla f(\mathbf{x})=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ and $C^\top\boldsymbol{\mu}=\begin{bmatrix}1\\1\\1\end{bmatrix}\mu$. So the stationarity condition is*

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}+\begin{bmatrix}1\\1\\1\end{bmatrix}\mu=\begin{bmatrix}x_1+\mu\\x_2+\mu\\x_3+\mu\end{bmatrix}=\mathbf{0}.$$

- *This can be solved as $x_1=-\mu$, $x_2=-\mu$, and $x_3=-\mu$.*

(d) Write down the feasibility constraints.

- *The feasibility condition is only* $C\mathbf{x} = \mathbf{d}$. *We don't need to worry about* $A\mathbf{x} = \mathbf{b}$ *or* $\boldsymbol{\lambda} \geq \mathbf{0}$ *since there are no inequality constraints.*
- *Here, that is*

$$x_1 + x_2 + x_3 = 3.$$

(e) Solve for the stationary point(s) by solving the stationarity and feasibility constraints together.

- *From before, we have* $x_1 = -\mu$, $x_2 = -\mu$, *and* $x_3 = -\mu$.
- *Plugging into* $x_1 + x_2 + x_3 = 3$, *this means* $-3\mu = 3$, *which implies* $\mu = -1$.
- *So the stationary point is* $\mathbf{x} = (1, 1, 1)$.

(f) Is the stationary point(s) optimal? Justify your answer.

The problem is convex since f *is a convex function and the constraints are linear. So the stationary point is a global optimal solution of the problem.*

Worksheet 10-1: Q3

Consider the problem

$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & x_1^2 + 2x_2^2 + 4x_1x_2 \\ \text{subject to} \quad & x_1 + x_2 = 1, \\ & x_1, x_2 \geq 0 \end{aligned}$$

.

(a) Is this problem covex? Justify your answer.

The problem is not convex because the objective function is not convex. To see this, we can compute the Hessian of the objective function, which is

$$H = \begin{bmatrix} 2 & 4 \\ 4 & 4 \end{bmatrix}.$$

The eigenvalues of H are -2 and 8, so H is not positive semidefinite. Therefore, the objective function is not convex, and the problem is not convex.

(b) Recall the Generalized Extreme Value Theorem:

Theorem 1 (GEVT). *If $f : U \to \mathbb{R}$ is a continuous function and $U \subseteq \mathbb{R}^n$ is compact, then f is bounded and there exists $\mathbf{x}^*, \mathbf{x}_* \in U$ such that $f(\mathbf{x}^*) = \sup_{\mathbf{x} \in U} f(\mathbf{x})$ and $f(\mathbf{x}_*) = \inf_{\mathbf{x} \in U} f(\mathbf{x})$.*

Use the GEVT to argue that there exists an optimal solution to the problem above.

- *The objective function is continuous since it is a polynomial.*
- *The feasible set is compact since it is closed and bounded. The feasible set is closed since it is the intersection of the closed sets $\{(x_1, x_2) : x_1 + x_2 = 1\}$, $\{(x_1, x_2) : x_1 \geq 0\}$, and $\{(x_1, x_2) : x_2 \geq 0\}$. The feasible set is bounded since $x_1 + x_2 = 1$ implies $x_1 \leq 1$ and $x_2 \leq 1$.*
- *Since the objective function is continuous and the feasible set is compact, the GEVT implies that there exists an optimal solution to the problem above.*

(c) Find the Lagrangian.

Following the notation above,

- $f(x_1, x_2) = x_1^2 + 2x_2^2 + 4x_1x_2$
- $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ *(since the inequality constraints are $-x_1 \leq 0$ and $-x_2 \leq 0$)*
- $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $\mathbf{d} = 1$ *(since the equality constraint is $x_1 + x_2 = 1$)*

So the Lagrangian is

$$L(x_1, x_2, \lambda_1, \lambda_2, \mu) = x_1^2 + 2x_2^2 + 4x_1x_2 + \lambda_1(-x_1) + \lambda_2(-x_2) + \mu(x_1 + x_2 - 1).$$

(d) Write down the stationarity KKT condition.

- *The stationarity condition is*

$$\begin{aligned}\nabla_{\mathbf{x}} L(\mathbf{x}, \mu) &= \nabla f(\mathbf{x}) + A^\top \boldsymbol{\lambda} + C^\top \boldsymbol{\mu} \\ &= \begin{bmatrix} 2x_1 + 4x_2 \\ 4x_1 + 4x_2 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}^\top \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mu = \mathbf{0}.\end{aligned}$$

- *With matrix multiplication, this is equivalent to the system of equations*

$$\begin{aligned} 2x_1 + 4x_2 - \lambda_1 + \mu &= 0, \\ 4x_1 + 4x_2 - \lambda_2 + \mu &= 0. \end{aligned}$$

(e) Write down the complementary slackness conditions.

- *The complementary slackness conditions are* $\lambda_i(\mathbf{a}_i^\top \mathbf{x}^* - b_i) = 0$ *for each inequality constraint* $\mathbf{a}_i^\top \mathbf{x} \le b_i$.
- *Here, the inequality constraints are* $-x_1 \le 0$ *and* $-x_2 \le 0$, *so the complementary slackness conditions are*

$$\begin{aligned} \lambda_1(-x_1) &= 0, \\ \lambda_2(-x_2) &= 0. \end{aligned}$$

- *This can be simplifed to* $\lambda_1 x_1 = 0$ *and* $\lambda_2 x_2 = 0$.

(f) Write down the feasibility conditions from the problem constraints.

- $x_1 + x_2 = 1$
- $x_1 \ge 0$
- $x_2 \ge 0$

(g) Write down the feasibility conditions for each of the λs.

- $\lambda_1 \ge 0$
- $\lambda_2 \ge 0$

(h) From everything above, copy down the 9 total equations/inequalities to be satisfied by an optimal solution.

$$\begin{aligned} 2x_1 + 4x_2 - \lambda_1 + \mu &= 0, \\ 4x_1 + 4x_2 - \lambda_2 + \mu &= 0, \\ \lambda_1 x_1 &= 0, \\ \lambda_2 x_2 &= 0, \\ x_1 + x_2 &= 1, \\ x_1 &\geq 0, \\ x_2 &\geq 0, \\ \lambda_1 &\geq 0, \\ \lambda_2 &\geq 0. \end{aligned}$$

(i) This problem involves multiple options for λ_1 and λ_2. We will address each of them separately.

  (i) Take the case $\lambda_1 = \lambda_2 = 0$. Use the slackness conditions to solve for x_1, and x_2. Is the solution consistent with the constraints?

  - *If* $\lambda_1 = \lambda_2 = 0$, *then three of the equations simplify to*

$$\begin{aligned} 2x_1 + 4x_2 + \mu &= 0, \\ 4x_1 + 4x_2 + \mu &= 0 \\ x_1 + x_2 &= 1. \end{aligned}$$

  - *Solving this system of equations gives* $x_1 = 0$, $x_2 = 1$, $\mu = -4$.
  - *This satisfies the remainder of the equations/inequalities, since* $x_1 \geq 0$, $x_2 \geq 0$, $\lambda_1 \geq 0$, *and* $\lambda_2 \geq 0$.
  - *So,* $(x_1, x_2) = (0, 1)$ *is a KKT point.*

  (ii) Take the case $\lambda_1, \lambda_2 > 0$. Use the slackness conditions to solve for x_1, and x_2. Is the solution consistent with the constraints?

  - *If* $\lambda_1, \lambda_2 > 0$, *then the slackness conditions* $\lambda_1 x_1 = 0$ *and* $\lambda_2 x_2 = 0$ *imply that* $x_1 = 0$ *and* $x_2 = 0$.
  - *But this contradicts the equality constraint* $x_1 + x_2 = 1$.
  - *So there is no KKT point arising from* $\lambda_1, \lambda_2 > 0$.

(iii) Take the case $\lambda_1 > 0, \lambda_2 = 0$. Use the slackness conditions to solve for x_1, and x_2. Is the solution consistent with the constraints?

- *If $\lambda_1 > 0$ and $\lambda_2 = 0$, then the slackness conditions $\lambda_1 x_1 = 0$ and $\lambda_2 x_2 = 0$ imply that $x_1 = 0$ but x_2 can be anything at this point.*
- *Plugging into the equality constraint $x_1 + x_2 = 1$, this means $x_2 = 1$.*
- *This satisfies the remainder of the equations/inequalities, since $x_1 \geq 0$, $x_2 \geq 0$, $\lambda_1 \geq 0$, and $\lambda_2 \geq 0$.*
- *So, $(x_1, x_2) = (0, 1)$ is a KKT point. However, note that we already found this one in part (i), so this is not a new KKT point.*

(iv) Take the case $\lambda_1 = 0, \lambda_2 > 0$. Use the slackness conditions to solve for x_1, and x_2. Is the solution consistent with the constraints?

- *If $\lambda_1 = 0$ and $\lambda_2 > 0$, then the slackness conditions $\lambda_1 x_1 = 0$ and $\lambda_2 x_2 = 0$ imply that $x_2 = 0$ but x_1 can be anything at this point.*
- *Plugging into the equality constraint $x_1 + x_2 = 1$, this means $x_1 = 1$.*
- *This satisfies the remainder of the equations/inequalities, since $x_1 \geq 0$, $x_2 \geq 0$, $\lambda_1 \geq 0$, and $\lambda_2 \geq 0$.*
- *So, $(x_1, x_2) = (1, 0)$ is a KKT point.*

(j) Write down the KKT points you found above (there should be two of them).

The KKT points are $(x_1, x_2) = (0, 1)$ *and* $(x_1, x_2) = (1, 0)$.

(k) Can you use the KKT theorem to determine which of the points you found above is a local optimal solution of the problem? Justify your answer.

No, since the problem is not convex, the KKT conditions are not sufficient for optimality. So we cannot use the KKT theorem to determine which of the points is a local optimal solution of the problem.

(l) Which of the two points is the global optimal solution of the problem? Justify your answer.

- *Even though we can't use the KKT condition theorem, we know that the GEVT guarantees that there is a global optimal solution to the problem.*
- *We also know from the KKT conditions that any local optimal solution must be a KKT point. So if we know there is a global optimal solution, it must be one of the KKT points.*
- *We can evaluate the objective function at each of the KKT points to determine which one is the global optimal solution.*
- *At* $(x_1, x_2) = (0, 1)$, *the objective function is* $0^2 + 2(1)^2 + 4(0)(1) = 2$.
- *At* $(x_1, x_2) = (1, 0)$, *the objective function is* $1^2 + 2(0)^2 + 4(1)(0) = 1$.
- *Since we are minimizing, the global optimal solution is* $(x_1, x_2) = (1, 0)$, *which has a smaller objective function value than* $(0, 1)$.