Convex Optimization: Part 1

Lecture 8-1 - CMSE 382

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This Lecture

Topics:

- Convex optimization definition
- Linear programming

Topics not covered but good videos to watch:

- Convex quadratic problems (including Classification via Linear Separators)
- Chebyshev center for a set of points

Announcements:

- Homework 3 due TODAY!
- The homework uses CVXPY. If you missed last class, make sure you get your CVXPY installation working ASAP!
- Next week: Homework 4 due Friday, March 20th. No quiz.

Section 1

Convex Optimization Definition

Convex Optimization: General Definition

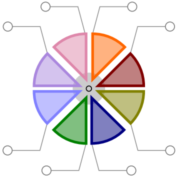
$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{such that} \quad & \mathbf{x} \in C \end{aligned}$$

where C is a convex set, and f is a convex function over C.

Smart Grid

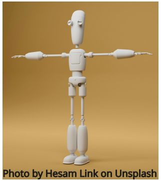
Minimize cost.
Control: Production, Demand, Transmission.

Portfolio Selection



Minimize risk.
Control Assets' allocation.

Robot balancing



Minimize tracking errors.
Control joint positions.

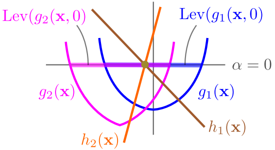
Functional definition

Convex Optimization: Functional Definition

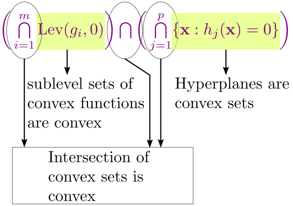
$$\begin{aligned} \text{minimize} \quad & f(\mathbf{x}) \\ \text{such that} \quad & g_i(\mathbf{x}) \leq 0, i = 1, 2, \ldots, m, \\ & h_j(\mathbf{x}) = 0, j = 1, 2, \ldots, p, \end{aligned}$$

where $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ are convex functions and $h_1, h_2, \ldots, h_p : \mathbb{R}^n \to \mathbb{R}$ are affine functions.

Example:



Generally:

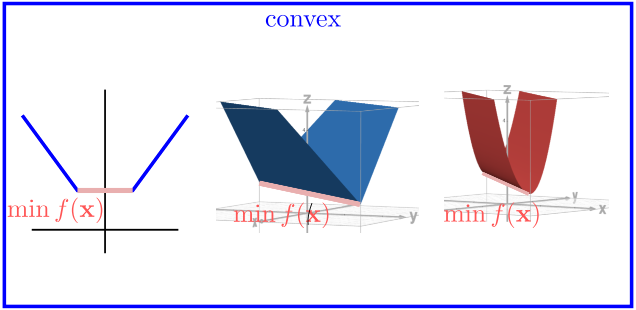


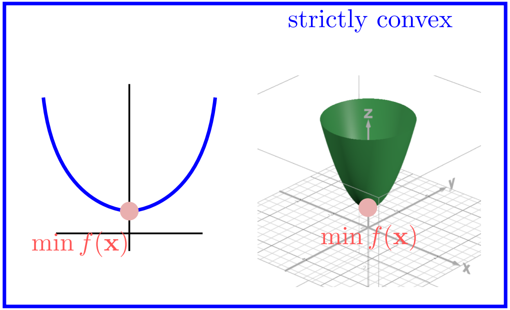
Reasons we like convex optimization: Local minima are global minima

Theorem (local minimum is global minimum in convex optimization)

Let $f : C \to \mathbb{R}$ be defined on the convex set C. Let $x^* \in C$ be a local minimum of f over C.

- *If f is* **convex**, *then $\mathbf{x}^*$ is a global minimum of f over C.*
- *If f is* **strictly convex**, *then $\mathbf{x}^*$ is a strict global minimum of f over C.*





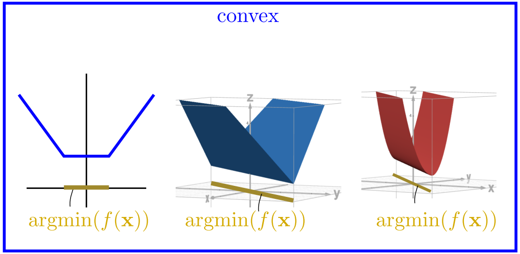
Theorem (Convexity of the optimal set in convex optimization)

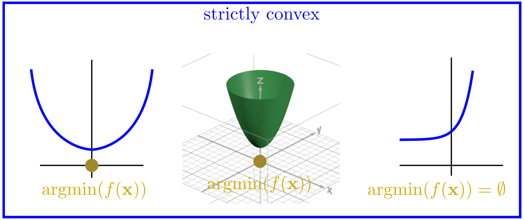
Let $f : C \to \mathbb{R}$ be a convex function defined over the convex set $C \subseteq \mathbb{R}^n$. Let X^* be the set of optimal solutions of the problem given by the equation

$$X^* = \mathrm{argmin}\{f(\mathbf{x}) : \mathbf{x} \in C\}.$$

Then:

- *If f is* **convex**, *then X^* is convex.*
- *If f is* **strictly convex**, *then X^* cotains at most one optimal solution.*





Section 2

Convex Optimization Example: Linear Programming

Find:

[Production(P),
Demand(D),
Transmission (T)]

That maximizes efficiency

$c_1 \times P + c_2 \times D + c_3 \times T$

Subject to constraints:

production = demand

transmission ≤ grid capacity

energy stored ≤ capacity

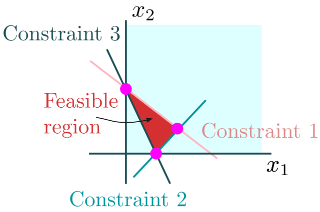
⋮

Linear optimization

$$\begin{aligned} \text{Find} \quad & \mathbf{x} \\ \min \quad & c^\top \mathbf{x} \\ \text{Subject to} \quad & A\mathbf{x} \leq \mathbf{b}; \\ & B\mathbf{x} = \mathbf{g} \\ & \mathbf{x} \geq 0 \end{aligned}$$

$$\begin{aligned} \text{Find} \quad & \mathbf{x} \\ \max \quad & c^\top \mathbf{x} \\ \text{Subject to} \quad & A\mathbf{x} \leq \mathbf{b}; \\ & \mathbf{x} \geq 0 \end{aligned}$$

(Standard form)



Optimize $f(x_1, x_2)$ **subject to:**

Constraint 1
$m_{11}x_1 + m_{12}x_2 \leq c_1$

$x_1 \geq 0$
$x_2 \geq 0$

Constraint 2
$m_{21}x_1 + m_{22}x_2 \geq c_2$

Constraint 3
$m_{31}x_1 + m_{32}x_2 \geq c_3$

Definition

A **Feasible region** of an optimization problem is the set of all possible points that satisfy the problem's constraints.

- It represents all the possible candidates for the optimization solution.

Recall: Guarantees related to linear optimization

Linear programming problems seek to optimize a linear function $f(\mathbf{x}) = \mathbf{c}^\top \mathbf{x}$, which is both convex and concave.

Theorem (Existence of maximizers at extreme points)

Let $f : C \to \mathbb{R}$ be a convex and continuous function over the convex and compact set $C \subseteq \mathbb{R}^n$. Then there exists at least one maximizer of f over C that is an extreme point of C.

Theorem (equivalence between extreme points and bfs)

Let $P = \mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$, where $A \in \mathbb{R}^{m \times n}$has linearly independent rows and $b \in \mathbb{R}^m$. Then $\bar{x}$ is a basic feasible solution of P if and only if it is an extreme point of P.

Theorem (Fundamental theorem of linear programming)

If the problem has an optimal solution, then it necessarily has an optimal basic feasible solution.

Groups - Round 4

Group 1
Michal, Joseph, Saitej, Dev

Group 2
Kyle, Dori, Shanze, Jack

Group 3
Noah, Daniel, Lora, Scott

Group 4
Lowell, Tianjian, Aidan, Anthony

Group 5
Abigail, Breena, Arjun, Luis

Group 6
Purvi, Atticus, Andrew, Vinod

Group 7
Yousif, Jay, Arya, Morgan

Group 8
Jonid, Jake, Dominic, Maye

Group 9
Alice, K M Tausif, Monirul Amin, Ha

Group 10
Jamie, Zheng, Aaron, Long

Group 11
Lauryn, Karen, Sanskaar, Braedon

Group 12
Sai, Brandon, Igor, Quang Minh