

Name:

Present group members:

**Worksheet 8-1: Q1**

For each problem:

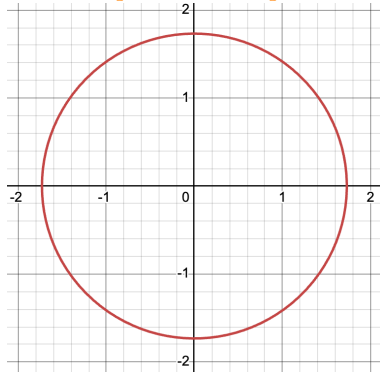
- Sketch the feasible set, is it convex?
- Is the objective function convex? You may use the atoms below.

$$\begin{array}{l|l} t^m & \text{convex for } m = 2, 4, 6 \\ \|\mathbf{x}\| & \text{convex} \\ e^t & \text{convex, nondecreasing} \end{array}$$

- Based on this, state whether the problem is a convex optimization problem or not.

(a) 
$$\begin{array}{ll} \min_{(x,y)} & x^2 - y \\ \text{s.t.} & x^2 + y^2 = 3. \end{array}$$

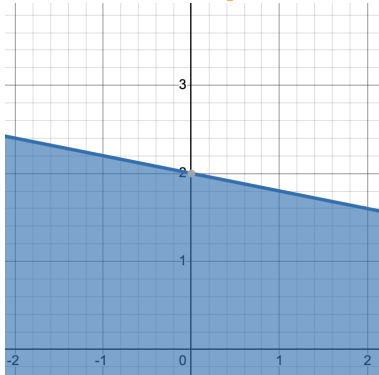
*The feasible set is the circle of radius  $\sqrt{3}$ , which is not a convex set. Therefore, this is not a convex optimization problem.*



*Note that the objective function is convex, but since the constraint is not convex this is not a convex optimization problem.*

(b) 
$$\begin{aligned} \min_{(x,y)} \quad & x + y^4 \\ \text{s.t.} \quad & x + 5y \leq 10. \end{aligned}$$

The feasible set is the half-space defined by  $x + 5y \leq 10$ , which is a convex set. The objective function is the sum of a linear function and a convex function, which is convex. Therefore, this is a convex optimization problem.



(c) 
$$\begin{aligned} \min_{(x,y)} \quad & -e^{x^2} + y^2 \\ \text{s.t.} \quad & \|(x, y)\| \leq 5. \end{aligned}$$

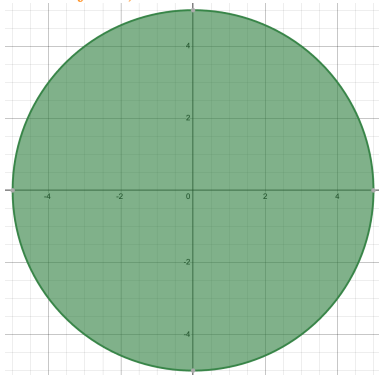
The feasible set is the ball of radius 5, which is a convex set. The objective function is the sum of a concave function and a convex function, which is not likely to be convex, however that's not enough to conclude that it's not a convex function.

Instead, we can check the Hessian of the objective function, which is given by

$$\begin{bmatrix} -2e^{x^2}(1 + 2x^2) & 0 \\ 0 & 2 \end{bmatrix}. \quad (1)$$

The eigenvalues of this Hessian are  $-2e^{x^2}(1 + 2x^2)$  and 2. Since the first one is negative and the second one is positive, the Hessian is not positive definite. Therefore, the objective function is not convex.

Therefore, this is not a convex optimization problem.



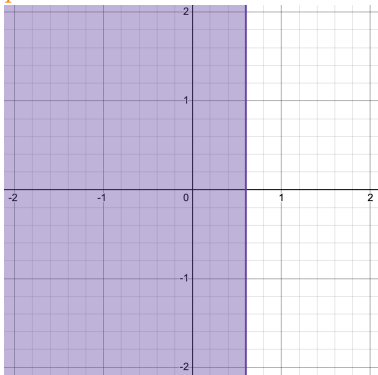
$$(d) \quad \min_{(x,y)} (1 + x^2 + x^{10})^2$$

$$\text{s.t.} \quad 5x + 1 \leq 4.$$

The feasible set is the half-space defined by  $5x + 1 \leq 4$ , which is a convex set. For the objective function, we need to build it up slowly.

- $x^2$  is convex.
- $x^{10}$  is convex.
- $1 + x^2 + x^{10}$  is the sum of a constant and two convex functions, which is convex.
- You can't just plug in a convex function (here  $1 + x^2 + x^{10}$ ) into another convex function (here  $(t)^2$ ) and expect it to be convex. However, we have a theorem that says that if we plug in a convex function into a **nondecreasing** convex function, then the resulting function is convex.
- $t^2$  is a nondecreasing function as long as we restrict our inputs to  $t \geq 0$ .
- For any input  $x \in \mathbb{R}$ , we have  $1 + x^2 + x^{10} \geq 1 > 0$ , so the input to the square function  $t^2$  is always non-negative.
- Thus the composition  $(1 + x^2 + x^{10})^2$  is convex.

The objective function is convex, and the constraint is convex, so this is a convex optimization problem.



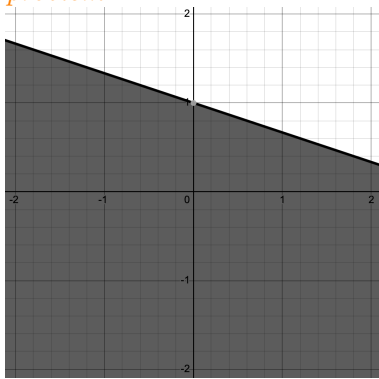
(e) 
$$\begin{aligned} \min_{(x,y)} \quad & -x^4 + 3y \\ \text{s.t.} \quad & x + 3y \leq 3. \end{aligned}$$

The feasible set is the half-space defined by  $x + 3y \leq 3$ , which is a convex set.

The objective function is the sum of a concave function and a linear function. The Hessian of the objective function is given by

$$\begin{bmatrix} -12x^2 & 0 \\ 0 & 0 \end{bmatrix}. \quad (2)$$

The eigenvalues of this Hessian are  $-12x^2$  and 0. Since the first one is negative and the second one is zero, the Hessian is not positive definite. Therefore, the objective function is not convex. Since the objective function is not convex, this is not a convex optimization problem.



### Worksheet 8-1: Q2

For a position in a company, we need to schedule job interviews for 3 candidates, Alice, Bob, and Charlie, who are coming in that order. The times that each candidate is available for an interview are given by the following table. The goal is to schedule the starting time for each candidate,

Candidate	Times	Interval as hours since 9:00am
Alice	9:00 - 10:30	[0, 1.5]
Bob	10:00 - 11:30	[1, 2.5]
Charlie	9:30 - 12:30	[0.5, 3.5]

denoted  $t_A$ ,  $t_B$ , and  $t_C$  for Alice, Bob, and Charlie respectively, such that the minimal starting time difference between consecutive interviews is maximal. *Note that in this problem we won't solve the problem, but we'll walk through how to get this problem into the standard form.*

- (a) The objective function is the minimal difference between consecutive starting times of interviews, which can be written as

$$f(t_A, t_B, t_C) = \min\{t_B - t_A, t_C - t_B\}.$$

Write the full optimization problem. *We want to maximize  $f(t_A, t_B, t_C)$  subject to the constraints on the availability of each candidate, which we can write as the intervals in the table above. So the full problem is*

$$\begin{aligned} \max_{t_A, t_B, t_C} \quad & \min\{t_B - t_A, t_C - t_B\} \\ \text{s.t.} \quad & 0 \leq t_A \leq 1.5, \\ & 1 \leq t_B \leq 2.5, \\ & 0.5 \leq t_C \leq 3.5. \end{aligned}$$

*Note that we want to have the candidates come in order, so we might want to add the constraints  $t_A \leq t_B \leq t_C$ . However, we can actually ignore this because an optimal solution will never have  $t_B < t_A$  or  $t_C < t_B$  since that would only increase the objective function.*

(b) We can pull a mathematical trick by introducing a new variable  $s$  to rewrite the optimization problem as

$$\begin{aligned} & \max_{t_A, t_B, t_C, s} && s \\ \text{s.t.} & && \min\{t_B - t_A, t_C - t_B\} \geq s, \\ & && \text{Insert your constraints from part (a) here.} \end{aligned}$$

Then we can break up the min function. Fill in the blanks below to rewrite the problem without the min function.

$$\begin{aligned} & \max_{t_A, t_B, t_C, s} && s \\ \text{s.t.} & && t_B - t_A \geq s, \\ & && t_C - t_B \geq s, \\ & && t_A \geq 0 \\ & && t_A \leq 1.5 \\ & && t_B \geq 1 \\ & && t_B \leq 2.5 \\ & && t_C \geq 0.5 \\ & && t_C \leq 3.5. \end{aligned}$$

Both of the first entries should be  $\geq$ , since the two together give us  $\min\{t_B - t_A, t_C - t_B\} \geq s$ .

(c) Use the above to write the optimization problem in the standard form

$$\begin{aligned} \max_{\mathbf{x}} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{s.t.} \quad & A\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{aligned}$$

by defining  $\mathbf{x}$ ,  $\mathbf{c}$ ,  $A$ , and  $\mathbf{b}$ .

- *The first thing to notice is that we want  $Ax \leq B$ , but we have some  $\geq$  constraints. We can multiply those by  $-1$  to get them in the right form. They also have variables on both side, so we can add one side to the other to get all variables together. In that version, the whole problem becomes*

$$\begin{aligned} \max_{t_A, t_B, t_C, s} \quad & s \\ \text{s.t.} \quad & t_A - t_B + s \leq 0, \\ & t_B - t_C + s \leq 0, \\ & -t_A \leq 0, \\ & t_A \leq 1.5, \\ & -t_B \leq -1, \\ & t_B \leq 2.5, \\ & -t_C \leq -0.5, \\ & t_C \leq 3.5. \end{aligned}$$

- *So we can set the following:*

$$\begin{aligned} - \mathbf{x} &= (t_A, t_B, t_C, s)^\top, \\ - \mathbf{c} &= (0, 0, 0, 1)^\top, \\ - A &= \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \text{ and} \\ - \mathbf{b} &= (0, 0, 0, 1.5, -1, 2.5, -0.5, 3.5)^\top. \end{aligned}$$

- *Make sure you understand how each row in  $A$  and each element in  $\mathbf{b}$  corresponds to a constraint in the problem.*

### Worksheet 8-1: Q3

Consider the following optimization problem:

$$\begin{aligned} \max \quad & 6x + 5y \\ \text{Subject to} \quad & 2x - 3y \leq 5 \\ & x + 4y \leq 11 \\ & 4x + y \leq 15 \\ & x, y \geq 0. \end{aligned}$$

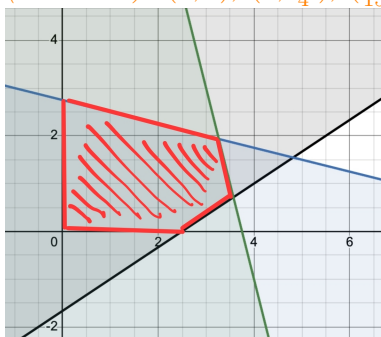
- (a) Find  $\mathbf{A}$ ,  $\mathbf{b}$ , and  $\mathbf{c}$  such that the problem can be written in the *standard form*  
*We have  $\mathbf{c} = (6, 5)^\top$ ,*

$$A = \begin{bmatrix} 2 & -3 \\ 1 & 4 \\ 4 & 1 \end{bmatrix},$$

*and  $\mathbf{b} = (5, 11, 15)^\top$ .*

- (b) Sketch the feasible set. Where are the extreme points of the feasible set?

*A sketch is below. The extreme points are the points where the lines intersect, which are (clockwise):  $(0, 0)$ ,  $(0, \frac{11}{4})$ ,  $(\frac{49}{15}, \frac{29}{15})$ ,  $(\frac{25}{7}, \frac{5}{7})$ , and  $(\frac{5}{2}, 0)$ .*



- (c) Solve the problem by finding the optimal solution.

*Checking the function value on each of the extreme points, we have*

Extreme point	Objective function value
$(0, 0)$	0
$(0, \frac{11}{4})$	13.75
$(\frac{49}{15}, \frac{29}{15})$	29.27
$(\frac{25}{7}, \frac{5}{7})$	25
$(\frac{5}{2}, 0)$	15

*The optimal solution is at  $(\frac{29}{7}, \frac{1}{7})$  with the optimal value  $\frac{179}{7}$ .*

### Worksheet 8-1: Q4

We are planning a football watch party for MSU by making pans of cookies and muffins. A pan of muffins sells for \$6 and a pan of cookies sells for \$10. In addition, we want to make at least as many pans of cookies as pans of muffins. We have only 13 cups of sugar, and a pan of muffins requires  $1/2$  a cup of sugar and a pan of cookies requires 1 cup of sugar. How many pans of each should we make to minimize the number of pans we need to bake? We want to make at least \$100. Let  $X$  be the number of pans of muffins and  $y$  be the number of pans of cookies. We want to solve this as a minimization problem.

(a) Write the variables and the objective function.

- **Variables:** Let  $x$  be the number of pans of muffins and  $y$  be the number of pans of cookies.
- **Objective function:** We want to minimize the number of pans we need to bake, which is given by  $x + y$ . So the objective is  $\min x + y$ .

(b) Write down the constraints as inequalities/equalities.

- We want to make at least \$100, which gives us the constraint  $6x + 10y \geq 100$ .
- We want to make at least as many pans of cookies as pans of muffins, which gives us the constraint  $y \geq x$ .
- We have only 13 cups of sugar, and a pan of muffins requires  $1/2$  a cup of sugar and a pan of cookies requires 1 cup of sugar, which gives us the constraint  $0.5x + y \leq 13$ .
- We can't make a negative number of pans, so we have the constraints  $x \geq 0$  and  $y \geq 0$ .

(c) Recast any inequalities and then write the problem in standard form.

- **Recast inequalities:**

$$-6x - 10y \leq -100$$

$$x - y \leq 0$$

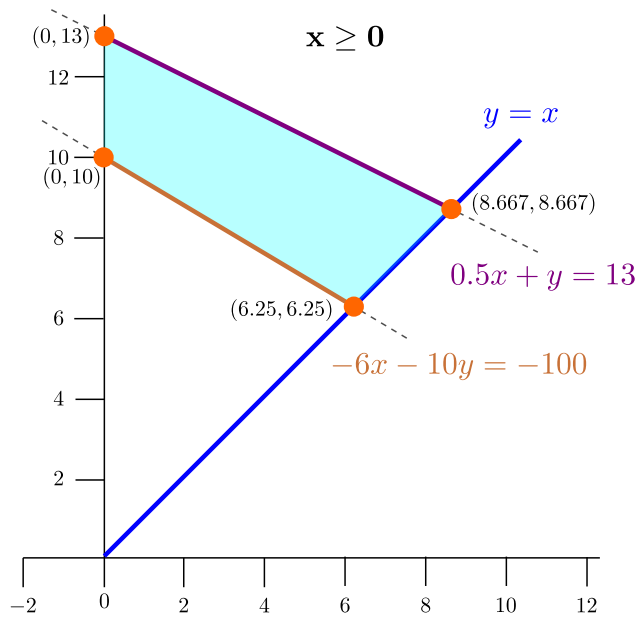
$$0.5x + y \leq 13$$

$$x, y \geq 0$$

- **Standard form:**

$$\begin{aligned} \min_{x,y} \quad & x + y \\ \text{s.t.} \quad & \begin{bmatrix} -6 & -10 \\ 1 & -1 \\ 0.5 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \leq \begin{bmatrix} -100 \\ 0 \\ 13 \end{bmatrix} \\ & \begin{bmatrix} x \\ y \end{bmatrix} \geq 0 \end{aligned}$$

(d) Sketch the feasible region.



(e) Find the minimum of the objective function.

*We need to check all the extreme points of the feasible region:*

<i>Extreme point</i>	<i>Objective function value</i>
$(0, 10)$	10
$(0, 13)$	13
$(6.25, 6.25)$	12.5
$(8.667, 8.667)$	17.334

*The minimum occurs at  $(x, y) = (0, 10)$  with the optimal value 10. If we must have both muffins and cookies, then the next best option is  $(x, y) = (6.25, 6.25)$  with the optimal value 12.5.*