Optimality Conditions for Linearly Constrained Problems

Lecture 10-2 - CMSE 382

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This Lecture

Topics (Review):

- KKT conditions
- Lagrangian function

Added topic:

- Active Constraints

Announcements:

- Quiz Weds April 1

Section 1

Optimality Conditions

Optimality conditions

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{such that} & \mathbf{x} \in C \end{array}$$

Linearly Constrained

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{such that} & \mathbf{a}_i^\top \mathbf{x} \leq b_i, \\ & i = 1, 2, \ldots, m. \end{array}$$

KKT Conditions

Inequality Constrained

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{such that} & g_i(\mathbf{x}) \leq 0, \\ & i = 1, 2, \ldots, m. \end{array}$$

KKT Conditions

Inequality & Equality Constrained

$$\begin{array}{ll} \text{minimize} & f(\mathbf{x}) \\ \text{such that} & g_i(\mathbf{x}) \leq 0, h_j(\mathbf{x}) = 0, \\ & i = 1, 2, \ldots, m. \\ & j = 1, 2, \ldots, p. \end{array}$$

KKT Conditions

KKT for linearly constrained problems

Theorem (Necessary optimality conditions)

Consider the minimization problem

$$(P) \quad \min_{\mathbf{x}} f(\mathbf{x})$$
$$\text{s.t.} \quad \mathbf{a}_i^T \mathbf{x} \leq b_i, \quad i = 1, 2, \ldots, m,$$

where f is continuously differentiable over $\mathbb{R}^n$, $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$, $b_1, b_2, \ldots, b_m \in \mathbb{R}$, and let $\mathbf{x}^*$ be a local minimum point of (P). Then there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ such that

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i a_i = 0, \quad \textit{and} \quad \lambda_i(\mathbf{a}_i^T \mathbf{x}^* - b_i) = 0, \quad i = 1, 2, \ldots, m.$$

- $\lambda_1, \ldots, \lambda_m$ are Lagrange multipliers. Non-negative for minimization with inequality constraints.

KKT for **convex** linearly constrained problems

Theorem (Necessary and sufficient optimality conditions)

Consider the minimization problem

$$(P) \quad \min_{\mathbf{x}} f(\mathbf{x}) \quad s.t. \quad \mathbf{a}_i^T \mathbf{x} \leq b_i, \quad i = 1, 2, \ldots, m,$$

where f is a **convex** *continuously differentiable over $\mathbb{R}^n$, $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$, $b_1, b_2, \ldots, b_m \in \mathbb{R}$, and let $\mathbf{x}^*$ be a feasible solution of (P). Then $\mathbf{x}^*$ is an optimal solution of (P)* **if and only if** *there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \geq 0$ such that*

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i a_i = 0, \quad \text{and} \quad \lambda_i(\mathbf{a}_i^T \mathbf{x}^* - b_i) = 0, \quad i = 1, 2, \ldots, m.$$

- The condition $\lambda_i(\mathbf{a}_i^T \mathbf{x}^* - b_i) = 0, \quad i = 1, 2, \ldots, m$ is called the complementary slackness condition.

The Lagrangian function

Definition (The Lagrangian function)

Consider the Nonlinear Programming Problem (NLP)

$$\text{(NLP)} \quad \min_{\mathbf{x}} f(\mathbf{x}) \quad \text{s.t.} \quad \{g_i(\mathbf{x}) \leq 0\}_{i=1}^{m}, \quad \{h_j(\mathbf{x}) = 0\}_{j=1}^{p},$$

where f, and all the g_i and h_j are continuously differentiable over $\mathbb{R}^n$.

The associated **Lagrangian function** is of the form

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \mu_j h_j(\mathbf{x}).$$

The **necessary KKT condition (stationarity condition)** is

$$\nabla_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \nabla f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}) + \sum_{j=1}^{p} \mu_j \nabla h_j(\mathbf{x}) = \mathbf{0}.$$

The Lagrangian function for linearly constrained optimization

Recall the minimization problem **with linear constraints**

$$(Q) \quad \min_{\mathbf{x}} f(\mathbf{x}) \quad \text{s.t.} \quad A\mathbf{x} \leq \mathbf{b}, \quad C\mathbf{x} = \mathbf{d}.$$

The associated **Lagrangian function** is of the form

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \boldsymbol{\lambda}^\top (A\mathbf{x} - \mathbf{b}) + \boldsymbol{\mu}^\top (C\mathbf{x} - \mathbf{d}).$$

The **necessary KKT condition** $\nabla f(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i \mathbf{a}_i + \sum_{j=1}^p \mu_j \mathbf{c}_j = \mathbf{0}$ can be written in terms of the Lagrangian as

$$\nabla_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \nabla f(\mathbf{x}) + A^\top \boldsymbol{\lambda} + C^T \boldsymbol{\mu} = \mathbf{0}.$$

Steps for finding the stationary points for a linearly constrained problem

- Write the problem in the standard form

$$\min_{\mathbf{x}} f(\mathbf{x}) \quad \text{s.t.} \quad A\mathbf{x} \leq \mathbf{b}, \quad C\mathbf{x} = \mathbf{d}.$$

- Write down the Lagrangian function

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \boldsymbol{\lambda}^\top (A\mathbf{x} - \mathbf{b}) + \boldsymbol{\mu}^\top (C\mathbf{x} - \mathbf{d}).$$

- Write down the KKT conditions

$$\nabla_{\mathbf{x}} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \nabla f(\mathbf{x}) + A^\top \boldsymbol{\lambda} + C^T \boldsymbol{\mu} = \mathbf{0}, \text{ and } \lambda_i(\mathbf{a}_i^\top \mathbf{x}^* - b_i) = 0.$$

- Write down the feasibility constraints

$$(A\mathbf{x} - \mathbf{b}) \quad \leq 0 \quad \text{and} \quad (C\mathbf{x} - \mathbf{d}) = 0$$

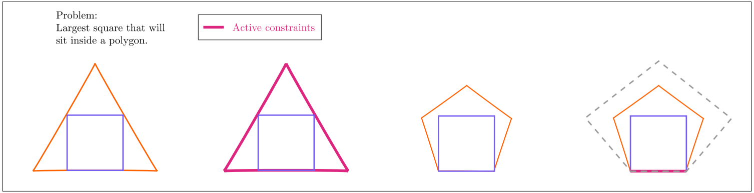
  - If inequality constraints are present, include $\boldsymbol{\lambda} \geq \mathbf{0}$ as a constraint.
- Solve the stationarity and feasibility constraints for the stationary points of the problem.
- If the problem is convex, then stationarity implies optimality.

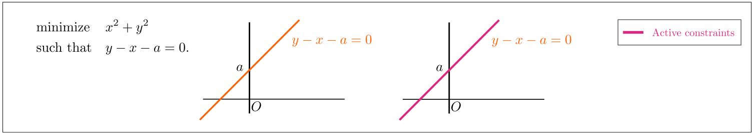
Section 2

Active Constraints

Optimality conditions

Active constraints





The constraint is **active** or **binding** when at the optimal solution the constraint holds with equality ''$=$'.

Video Example

$$\begin{aligned} \min \quad & x + y \\ \text{s.t.} \quad & -6x - 10y \leq -100 \\ & x - y \leq 0 \\ & 0.5x + y \leq 13 \\ & -x, -y \leq 0, \end{aligned}$$

Variables

- x, y
- $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5$

Stationarity

$$1 - 6\lambda_1 + \lambda_2 + 0.5\lambda_3 - \lambda_4 = 0$$
$$1 - 10\lambda_1 - \lambda_2 + \lambda_3 - \lambda_5 = 0$$

Slackness

$$\lambda_1(-6x - 10y + 100) = 0$$
$$\lambda_2(x - y) = 0$$
$$\lambda_3(0.5x + y - 13) = 0$$
$$\lambda_4(-x) = 0$$
$$\lambda_5(-y) = 0$$

Feasibility

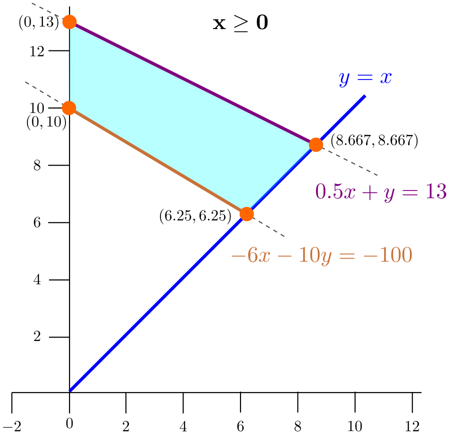
$$\lambda_i \geq 0$$
$$(i = 1, \ldots, 5)$$
$$-6x - 10y \leq -100$$
$$x - y \leq 0$$
$$0.5x + y \leq 13$$
$$-x, -y \leq 0.$$

Optimality conditions

Slackness

$$\lambda_1(-6x - 10y + 100) = 0$$
$$\lambda_2(x - y) = 0$$
$$\lambda_3(0.5x + y - 13) = 0$$
$$\lambda_4(-x) = 0$$
$$\lambda_5(-y) = 0$$

Optimal solution at $(x, y) = (0, 10)$.



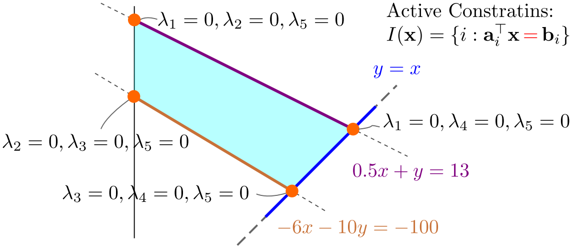
Active constraints for example

Which of the constraints is active or binding at the optimal solution?

$$-6x - 10y + 100 = 0$$

$$x = 0$$

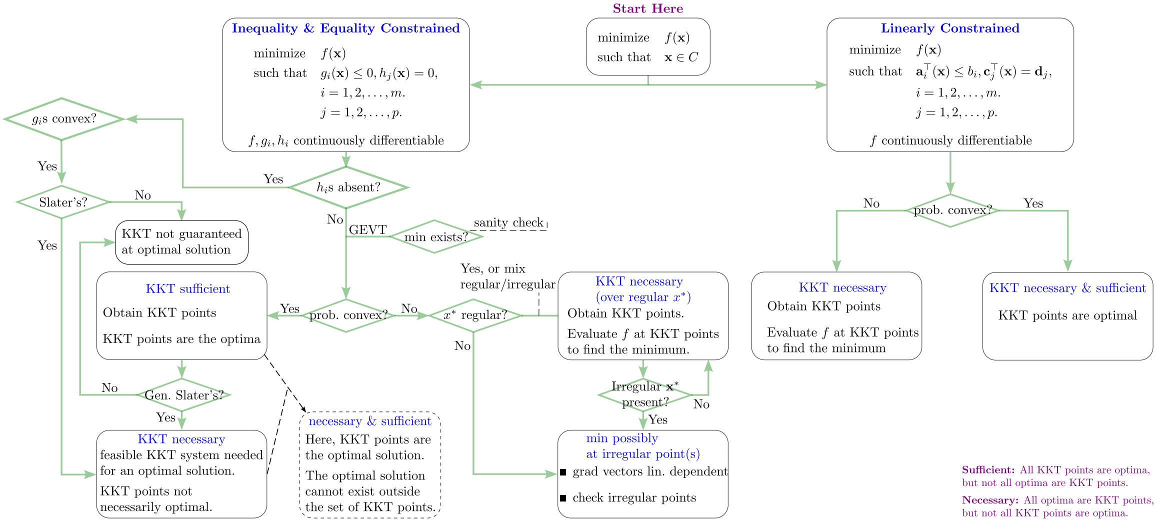
$\lambda_i = 0$ if i is not active.



Section 3

Big picture

Big picture view



Sufficient: All KKT points are optima, but not all optima are KKT points.

Necessary: All optima are KKT points, but not all KKT points are optima.

Groups - Round 5

Group 1
Michal, Kyle, Daniel, Purvi

Group 2
Joseph, Jack, Scott, Breena

Group 3
Saitej, Dori, Noah, Tianjian

Group 4
Dev, Shanze, Lowell, Andrew

Group 5
Lora, Aidan, Arjun, Monirul Amin

Group 6
Anthony, Abigail, Atticus, Yousif

Group 7
Luis, Vinod, Morgan, Dominic

Group 8
Jay, Jonid, Alice, Aaron

Group 9
Arya, Jake, K M Tausif, Lauryn

Group 10
Maye, Ha, Zheng, Sai

Group 11
Jamie, Karen, Brandon, Quang Minh

Group 12
Long, Sanskaar, Braedon, Igor