Convex Sets: Part 2

Lecture 6-2 - CMSE 382

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This Lecture

Topics:

- Convex cones
- Conic combinations

Announcements:

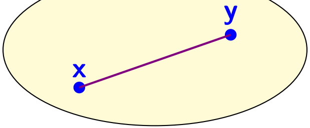
- Quiz wednesday

Last time

Definition

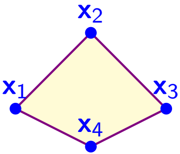
A set $C \subseteq \mathbb{R}^n$ is **convex** if for any $\mathbf{x}, \mathbf{y} \in C$, the line segment $[\mathbf{x}, \mathbf{y}]$ is also in C.

$$[\mathbf{x},\mathbf{y}] = \{\lambda \mathbf{x} + (1-\lambda)\mathbf{y} : 0 \leq \lambda \leq 1\}$$



Definition

The convex hull of a set $S \subseteq \mathbb{R}^n$ is the smallest convex set containing S.



Definition

A convex combination of points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ is a point of the form $\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \ldots + \lambda_k \mathbf{x}_k$, where $\lambda_i \geq 0$ and $\sum_{i=1}^k \lambda_i = 1$.

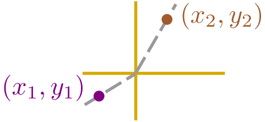
Section 1

Cones

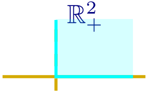
Definition

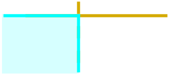
A set S is a **cone** if for any $\mathbf{x} \in S$ and $\lambda \geq 0$, $\lambda \mathbf{x} \in S$.

- **Union of cones is a cone**.
- **Intersection of cones is a cone**.
- (Remember that the union of convex sets is not necessarily convex, but the intersection of convex sets is convex)

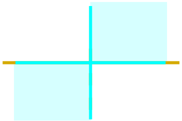


$$\left\{ m \begin{bmatrix} x \\ y \end{bmatrix} : m \geq 0, \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$





$$\{(x, y) | x \leq 0, y \leq 0\}$$



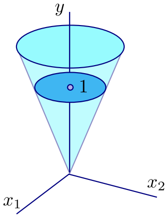
$$\mathbb{R}^2_+ \cup \{(x, y) | x \leq 0, y \leq 0\}$$

Theorem

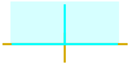
A cone S is convex if and only if $\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S$.

- Recall: A set S is a **cone** if for any $\mathbf{x} \in S$ and $\lambda \geq 0$, $\lambda \mathbf{x} \in S$.

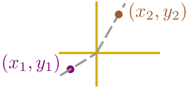
Convex Cones



$$L^2 = \left\{ \begin{bmatrix} \mathbf{x} \\ t \end{bmatrix} \Big| \|\mathbf{x}\| \leq t, \mathbf{x} \in \mathbb{R}^2, t \in \mathbb{R} \right\}$$

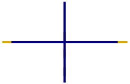


$\{(x,y) | x \in \mathbb{R}, y \geq 0\}$

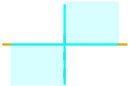


$$\left\{ m \begin{bmatrix} x \\ y \end{bmatrix} : m \geq 0, \begin{bmatrix} x \\ y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

Non-Convex Cones



x-axis $\cup$ y-axis



$\mathbb{R}^2_+ \cup \{(x,y) | x \leq 0, y \leq 0\}$

Section 2

Conic Combinations

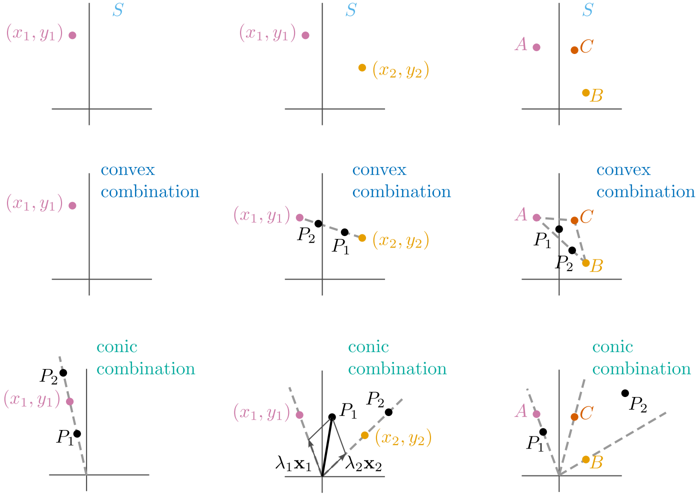
Conic combinations

Definition

Given k points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathbb{R}^n$, a **conic combination** of these k points is a vector of the form $\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_k\mathbf{x}_k$, where $\lambda_i \geq 0$.

- This is different than a convex combination because the λ_i do not need to sum to 1.

Conic combination examples



Conic hull

Definition

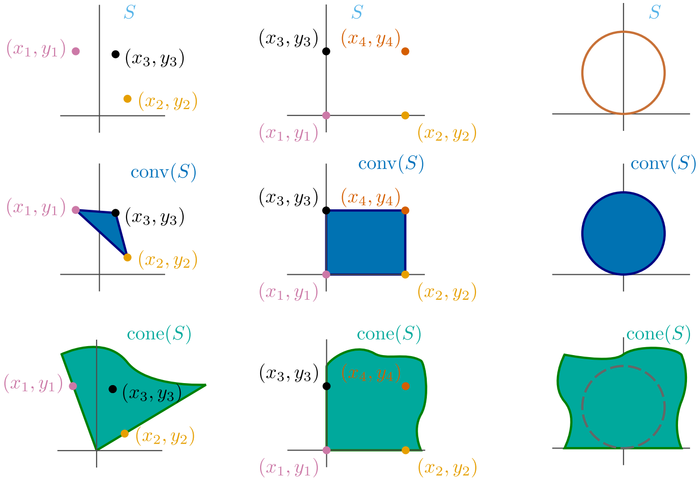
Let $S \subseteq \mathbb{R}^n$. Then the **conic hull** of S, denoted by $\mathrm{cone}(S)$, is the set of all conic combinations of vectors from S:

$$\mathrm{cone}(S) = \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{x}_i \mid \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in S, \lambda_i \geq 0, k \in \mathbb{N} \right\}$$

- The **conic hull** of a set S is the smallest convex cone containing S.
- For $S \subset \mathbb{R}^n$, $\mathrm{conv}(S) \subseteq \mathrm{cone}(S)$.

Convex sets

Conic hull examples



Conic Representation Theorem

Theorem

Let $S \subseteq \mathbb{R}^n$ and let $\mathbf{x} \in \mathrm{cone}(S)$. Then there exist k linearly independent vectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in S$ such that $\mathbf{x} \in \mathrm{cone}(\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k\})$; that is, there exists $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ such that

$$\mathbf{x} = \sum_{i=1}^{k} \lambda_i \mathbf{x}_i$$

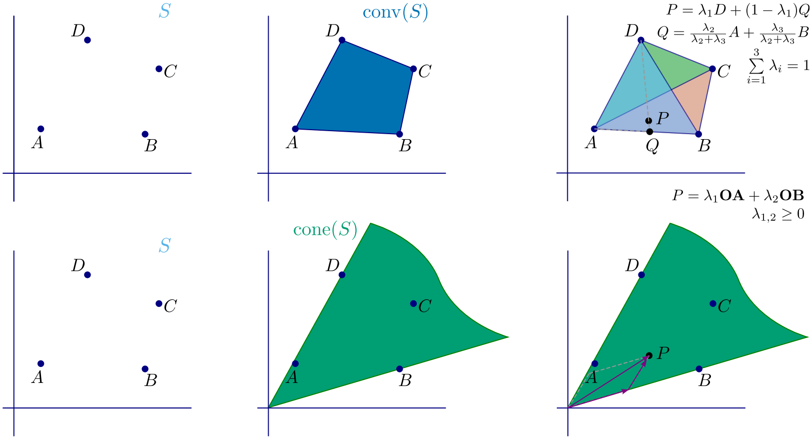
In addition, $k \leq n$.

- This says that each vector in the conic hull of a set $S \subseteq \mathbb{R}^n$ can be represented as a convex combination of at most n vectors from S.

Conic representation theorem examples

If $\mathbf{x} \in \mathrm{cone}(S)$, then $\mathbf{x}$ is the nonnegative sum of at most n points of S.

If $\mathbf{x} \in \mathrm{conv}(S)$, then $\mathbf{x}$ is the convex sum of at most $n+1$ points of S.



Conic combination versus convex combination

For $S \subset \mathbb{R}^n$:

A set S is a **cone** if for any $\mathbf{x} \in S$ and $\lambda \geq 0$, $\lambda \mathbf{x} \in S$.

Conic combination

$\lambda_1 \mathbf{x}_1 + \ldots + \lambda_n \mathbf{x}_n$

$\lambda_i \geq 0$

If $\mathbf{x} \in \mathrm{cone}(S)$, then $\mathbf{x}$ is the nonnegative sum of at most n points of S.

$\mathrm{cone}(S)$ is the smallest convex cone containing S.

A set $S \subseteq \mathbb{R}^n$ is **convex** if for any $\mathbf{x}, \mathbf{y} \in S$, the line segment $[\mathbf{x}, \mathbf{y}]$ is also in S.

Convex combination

$\lambda_1 \mathbf{x}_1 + \ldots + \lambda_n \mathbf{x}_n$

$\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$

If $\mathbf{x} \in \mathrm{conv}(S)$, then $\mathbf{x}$ is the convex sum of at most $n+1$ points of S.

$\mathrm{conv}(S)$ is the smallest convex set containing S.

Groups - Round 3

Group 1
Lowell, Tianjuan, Lauryn, Atticus

Group 2
Alice, Aidan, Dev, Anthony

Group 3
Abigail, Michal, Breena, Andrew

Group 4
Kyle, Vinod, Dori, Joseph

Group 5
Yousif, Jamie, Jay, K.M Tausif

Group 6
Shanze, Saitej, Karen, Jack

Group 7
Arjun, Noah, Luis, Arya

Group 8
Morgan, Jonid, Sanskaar, Jake

Group 9
Quang Minh, Monirul Amin, Daniel, Ha

Group 10
Braedon, Dominic, Zheng, Lora

Group 11
Sai, Brandon, Purvi, Aaron

Group 12
Igor, Scott, Maye, Long