Name: **Present group members:**

Worksheet 2-3: Q1

We are going to investigate the function $f(x,y) = 2x^2 - 8xy + y^2$.

1. Write the function in the quadratic form

$$f(x,y) = \mathbf{x}^\top A\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + c$$

for some symmetric matrix A, vector $\mathbf{b}$, and scalar c.

- Line up the function with the quadratic form definition.

$$\begin{aligned} f(x_1,x_2) &= [x_1 x_2] \begin{bmatrix} a_1 & a_2 \\ a_2 & a_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2 \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + c \\ &= a_1 x_1^2 + 2a_2 x_1 x_2 + a_4 x_2^2 + 2b_1 x_1 + 2b_2 x_2 + c \end{aligned}$$

- So $A = \begin{bmatrix} 2 & -4 \\ -4 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $c = 0$.

2. What is the gradient $\nabla f(x,y)$ and Hessian $\nabla^2 f(x,y)$ of the function? *(Hint: use your matrix A from above.)*

- $\nabla f = 2A\mathbf{x} + 2\mathbf{b} = 2 \begin{bmatrix} 2 & -4 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4x - 8y \\ -8x + 2y \end{bmatrix}$
- $\nabla^2 f = 2A = 2 \begin{bmatrix} 2 & -4 \\ -4 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -8 \\ -8 & 2 \end{bmatrix}$

3. Using this, is f coercive? Why or why not? *Since A is not positive definite (its eigenvalues are 5.53 and -2.53), the function is not coercive.*

4. Is f convex? Why or why not? *Since the Hessian is not positive semidefinite (its eigenvalues are 11.06 and -5.06), the function is not convex.*

5. Find and classify the stationary points of $f(x, y) = 2x^2 - 8xy + y^2$. If there are local optima, are they also global?

   - Set the gradient to zero and solve:

$$\begin{aligned} 4x - 8y &= 0 \\ -8x + 2y &= 0 \end{aligned}$$

     Solving this system gives the stationary point at $(0, 0)$.

   - The Hessian at (0,0), $\begin{bmatrix} 4 & -8 \\ -8 & 2 \end{bmatrix}$, is indefinite (eigenvalues 11.06 and -5.06), so the stationary point is a saddle point.
   - Since the function is not coercive and has no local minima, there are no global minima.

Worksheet 2-3: Q2
Now consider the function $g(x, y) = 2x^2 - 2xy + y^2 + 6x + 2y$.

1. Write the function in the quadratic form

$$g(x, y) = \mathbf{x}^\top A\mathbf{x} + 2\mathbf{b}^\top \mathbf{x} + c$$

for some symmetric matrix A, vector $\mathbf{b}$, and scalar c.

- Line up the function with the quadratic form definition.

$$\begin{aligned} f(x_1, x_2) &= [x_1 x_2] \begin{bmatrix} a_1 & a_2 \\ a_2 & a_4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2 \begin{bmatrix} b_1 & b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + c \\ &= a_1 x_1^2 + 2a_2 x_1 x_2 + a_4 x_2^2 + 2b_1 x_1 + 2b_2 x_2 + c \end{aligned}$$

- So $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, and $c = 0$.

2. What is the gradient $\nabla f(x, y)$ and Hessian $\nabla^2 f(x, y)$ of the function? *(Hint: use your matrix A from above.)*

- $\nabla f = 2A\mathbf{x} + 2\mathbf{b} = 2 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4x - 2y + 6 \\ -2x + 2y + 2 \end{bmatrix}$
- $\nabla^2 f = 2A = 2 \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$

3. Using this, is g coercive? Why or why not? *Since A is positive definite (its eigenvalues are 2.62 and 0.38), the function is coercive.*

4. Is g convex? Why or why not? *Since the Hessian is positive semidefinite (its eigenvalues are 5.23 and 0.76), the function is convex.*

5. Find and classify the stationary points of $g(x, y) = 2x^2 - 2xy + y^2 + 6x + 2y$. Use the properties of quadratic functions to determine if any local optima are also global optima.

- Set the gradient to zero and solve:

$$\begin{aligned} 4x - 2y + 6 &= 0 \\ -2x + 2y + 2 &= 0 \end{aligned}$$

  Solving this system gives the stationary point at $(-4, -5)$.

- The Hessian at (-4,-5), $\begin{bmatrix} 4 & -2 \\ -2 & 2 \end{bmatrix}$, is positive definite (eigenvalues $3 \pm \sqrt{5} > 0$), so the stationary point is a local minimum.

- Since the function is coercive and has a local minimum, this local minimum is also a global minimum.