Convex Sets: Part 3

Lecture 6-3 - CMSE 382

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This Lecture

Topics:

- Convex polytope
- Feasible region
- Basic feasible solutions
- Extreme points

Announcements:

- Quiz today!

Last time

Definition

A set $C \subseteq \mathbb{R}^n$ is **convex** if for any $\mathbf{x}, \mathbf{y} \in C$, the line segment $[\mathbf{x}, \mathbf{y}]$ is also in C.

Definition

A set $C \subseteq \mathbb{R}^n$ is a **cone** if for any $\mathbf{x} \in C$ and $\lambda \geq 0$, we have $\lambda \mathbf{x} \in C$.

Theorem

A set C is a **convex cone** *if and only if for any $\mathbf{x}, \mathbf{y} \in C$ and we have $\mathbf{x} + \mathbf{y} \in C$.*

Motivation

Linear Optimization (Linear Programming)

Linear optimization

Find	$\mathbf{x}$
That maximizes	$c^\top \mathbf{x}$
Subject to	$A\mathbf{x} \leq \mathbf{b}$;
	$\mathbf{x} \geq 0$

→

Standard form $A\mathbf{x} = \mathbf{b}$, $\mathbf{x} \geq \mathbf{0}$.
Rows of A are linearly indepedent.

Find basic feasible solutions (bfs)

→

When $\mathbf{x}$ is optimal
it will be at a
Basic Feasible Solutions (bfs)

↕

Extreme points

Section 1

Feasible Region

Polytopes

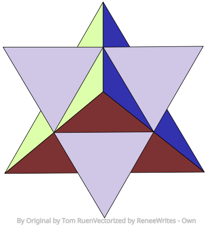
Definition

Polytope is a geoemtric object with flat faces.

- It generalizes polyhedra to higher dimensions.
  - Polyhedron is a 3-polytope.
  - Polygon is a 2-polytope.

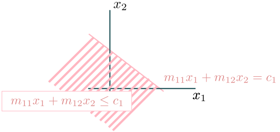
Definition

A polyhedron is a three-dimensional figure with flat polygonal faces, straight edges and sharp corners or vertices.

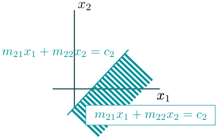


Definition

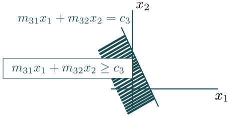
We define a **convex polytope** as the set $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} \leq \mathbf{b}\}$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.



Convex



Convex



Convex

Linear Inequalities

$m_{11}x_1 + m_{12}x_2 \leq c_1$

$m_{21}x_1 + m_{22}x_2 \leq c_2$

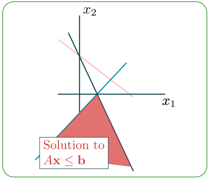
$m_{31}x_1 + m_{32}x_2 \leq c_3$

Matrix Form

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \leq \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

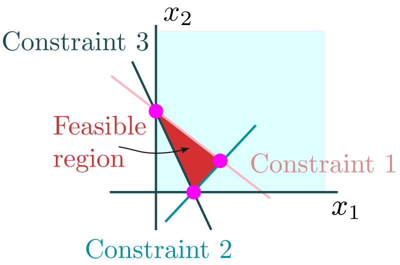
Compact Form

$$A\mathbf{x} \leq \mathbf{b}$$



Intersection of convex sets is convex

Feasible region



Optimize $f(x_1, x_2)$ **subject to:**

Constraint 1
$m_{11}x_1 + m_{12}x_2 \leq c_1$

$x_1 \geq 0$
$x_2 \geq 0$

Constraint 2
$m_{21}x_1 + m_{22}x_2 \geq c_2$

Constraint 3
$m_{31}x_1 + m_{32}x_2 \geq c_3$

Definition

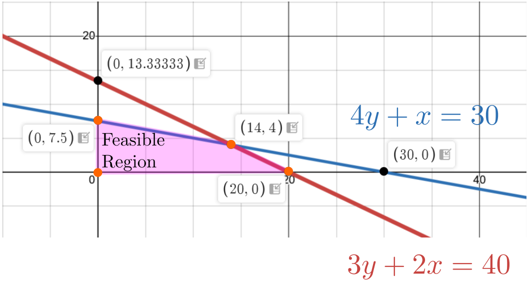
The **feasible region** of an optimization problem is the set of all possible points that satisfy the problem's constraints.

- It represents all the possible candidates for the optimization solution.

Example feasible region - 2D

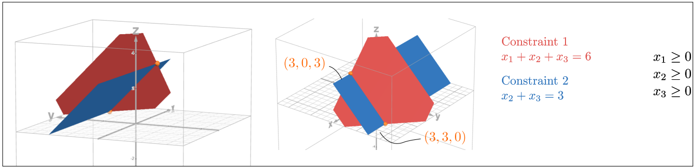
What is the feasible region for the linear system

$$3y + 2x \leq 40;$$
$$4y + x \leq 30;$$
$$x, y \geq 0.$$



What is the feasible region for the linear system

$$x_1 + x_2 + x_3 = 6;$$
$$x_2 + x_3 = 3;$$
$$x_1, x_2, x_3 \geq 0.$$



Section 2

Basic Feasible Solutions

Basic Feasible Solutions

Definition

Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax}=\mathbf{b}, x \geq 0\}$, where $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, and A's rows are linearly independent.

$\bar{\mathbf{x}}$ is a **basic feasible solution (bfs)** of P if the columns of $\mathbf{A}$ corresponding to the indices of the positive values of $\bar{\mathbf{x}}$ are linearly independent.

- If P is non-empty, then it contains at least one bfs.
  - This is shown by using the conic representation theorem from last class.
- A bfs has at most m non-zero elements.

Example

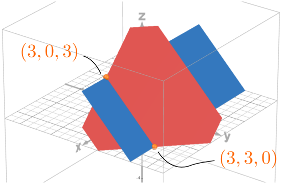
$$
\begin{aligned}
x_1 + x_2 + x_3 &= 6 \\
x_2 + x_3 &= 3 \\
x_1, x_2, x_3 &\geq 0.
\end{aligned}
$$

- Is $(x_1, x_2, x_3) = (3, 3, 0)$ a solution? Yes because $3 + 3 + 0 = 6$; $3 + 0 = 3$; $3, 3, 0 \geq 0$.
- Is it a bfs? Yes because the columns of A corresponding to the indices of the positive values of $\bar{\mathbf{x}}$ (which are x_1 and x_2) are linearly independent.

Linearly independent

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} 3 > 0 \\ 3 > 0 \\ 0 \end{bmatrix}$$



Constraint 1
$x_1 + x_2 + x_3 = 6$

Constraint 2
$x_2 + x_3 = 3$

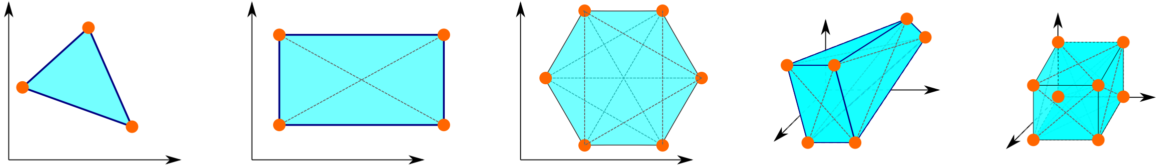
$x_1 \geq 0$
$x_2 \geq 0$
$x_3 \geq 0$

Extreme points

Definition

Let S be a convex set. A point $\mathbf{x} \in S$ is an **extreme point** of S if there do not exist two distinct points $\mathbf{x}_1, \mathbf{x}_2 \in S$ and $\lambda \in (0,1)$ such that $\mathbf{x} = \lambda \mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$.

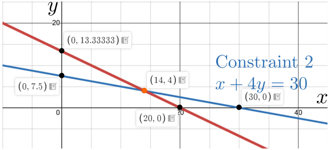
- It is a point in S that cannot be represented as a nontrivial convex combination of two different points in S.
- The set of all extreme points is denoted $\text{ext}(S)$.



Equivalence of extreme points and bfs

Theorem

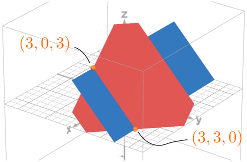
Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$, where $\mathbf{A} \in \mathbb{R}^{m \times n}$ has linearly independent rows and $\mathbf{b} \in \mathbb{R}^m$. Then $\bar{\mathbf{x}}$ is a basic feasible solution of P if and only if it is an extreme point of P.



Constraint 1
$2x + 3y = 40$

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 30 \\ 40 \end{bmatrix}$$

$$\bar{\mathbf{x}} = \begin{bmatrix} 14 \\ 4 \end{bmatrix}$$



Constraint 1
$x_1 + x_2 + x_3 = 6$

Constraint 2
$x_2 + x_3 = 3$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}$$

(a) $\bar{\mathbf{x}} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$ (b) $\bar{\mathbf{x}} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix}$

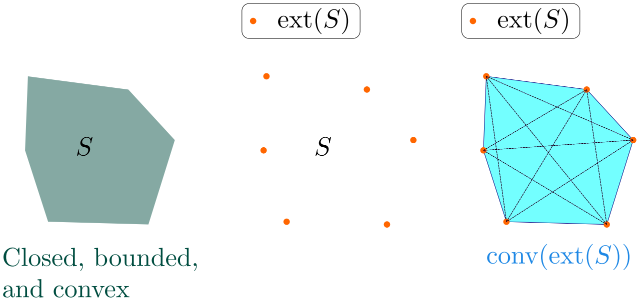
Extreme points

Theorem

Theorem: *Let $S \subseteq \mathbb{R}^n$ be a closed and bounded convex set. Then*

$$S = conv(ext(S))$$

- *A compact convex set is the convex hull of its extreme points.*



Groups - Round 3

Group 1
Lowell, Tianjuan, Lauryn, Atticus

Group 2
Alice, Aidan, Dev, Anthony

Group 3
Abigail, Michal, Breena, Andrew

Group 4
Kyle, Vinod, Dori, Joseph

Group 5
Yousif, Jamie, Jay, K.M Tausif

Group 6
Shanze, Saitej, Karen, Jack

Group 7
Arjun, Noah, Luis, Arya

Group 8
Morgan, Jonid, Sanskaar, Jake

Group 9
Quang Minh, Monirul Amin, Daniel, Ha

Group 10
Braedon, Dominic, Zheng, Lora

Group 11
Sai, Brandon, Purvi, Aaron

Group 12
Igor, Scott, Maye, Long