

Name:

Present group members:

Worksheet 6-3: Q1

Consider the optimization problem $Ax = b$ given by

$$\begin{bmatrix} 1 & 5 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 14 \\ 7 \end{bmatrix}$$

- (a) Which subsets of 2 columns of the matrix A are linearly independent? (Hint: there are 10 possible pairs.)

The linearly independent subsets of columns are: $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{1, 5\}$, $\{2, 3\}$, $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$, and $\{4, 5\}$.

Maybe more helpful is the pair that is not linearly independent: $\{3, 5\}$, since column 5 is just 2 times column 3.

- (b) Consider the point $\mathbf{x} = (0, 2, 0, 1, 0)$. Check that this a solution to the optimization problem $Ax = b$.

We compute Ax :

$$\begin{bmatrix} 1 & 5 & 3 & 4 & 6 \\ 0 & 1 & 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \cdot 2 + 4 \cdot 1 \\ 1 \cdot 2 + 5 \cdot 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 7 \end{bmatrix}$$

So, yes, $\mathbf{x} = (0, 2, 0, 1, 0)$ is a solution to $Ax = b$.

- (c) Is the point $\mathbf{x} = (0, 2, 0, 1, 0)$ a basic feasible solution? Why or why not?

Yes, $\mathbf{x} = (0, 2, 0, 1, 0)$ is a basic feasible solution. This is because the columns of A corresponding to the nonzero entries of \mathbf{x} (columns 2 and 4) are linearly independent, and \mathbf{x} is a solution to $Ax = b$.

Worksheet 6-3: Q2

Consider the optimization problem $Ax = b$ given by

$$\begin{bmatrix} 1 & 7 & 22 & 0 & 5 & 0 & 1 \\ 0 & 3 & -5 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & -4 & 0 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix}$$

- (a) Check that the set of the 1st, 4th, and 6th columns of A are linearly independent.
These are the standard basis for \mathbb{R}^3 , so they are linearly independent.
- (b) Find a solution to the optimization problem $Ax = b$ that corresponds to the 1st, 4th, and 6th columns of A . That is, find a solution such that the only nonzero entries of \mathbf{x} are the 1st, 4th, and 6th entries. Is it a basic feasible solution?

We want to find $\mathbf{x} = (x_1, 0, 0, x_4, 0, x_6, 0)$ such that $Ax = b$. We compute Ax :

$$\begin{bmatrix} 1 & 7 & 22 & 0 & 5 & 0 & 1 \\ 0 & 3 & -5 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & -4 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \\ 0 \\ x_6 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_6 \\ x_4 \end{bmatrix}$$

We want this to equal $\begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix}$, so we need $x_1 = 7$, $x_6 = 1$, and $x_4 = 2$. So, all the entries are positive corresponding to the 1st, 4th, and 6th columns of A , so a basic feasible solution is $\mathbf{x} = (7, 0, 0, 2, 0, 1, 0)$.

For the same optimization problem,

$$\begin{bmatrix} 1 & 7 & 22 & 0 & 5 & 0 & 1 \\ 0 & 3 & -5 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & -4 & 0 & 3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix}$$

- (c) Check that the 1st, 3rd, and 4th columns of A are linearly independent. *They are linearly independent because none can be written as a sum of the other two. For example, column 3 cannot be written as a sum of columns 1 and 4 because the middle entry of column 3 is -5, while the middle entries of columns 1 and 4 are both 0, and I can't add up multiples of 0 to get something that isn't zero.*
- (d) Find a solution to the optimization problem $Ax = b$ that corresponds to the 1st, 3rd, and 4th columns of A . That is, find a solution such that the only nonzero entries of \mathbf{x} are the 1st, 3rd, and 4th entries. Is it a basic feasible solution? Why or why not?

We want to find $\mathbf{x} = (x_1, 0, x_3, x_4, 0, 0, 0)$ such that $Ax = b$. We compute Ax :

$$\begin{bmatrix} 1 & 7 & 22 & 0 & 5 & 0 & 1 \\ 0 & 3 & -5 & 0 & 2 & 1 & 2 \\ 0 & 1 & 0 & 1 & -4 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ 0 \\ x_3 \\ x_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 + 22x_3 \\ -5x_3 \\ x_4 \end{bmatrix}$$

We want this to equal $\begin{bmatrix} 7 \\ 1 \\ 2 \end{bmatrix}$, so we need $x_3 = -\frac{1}{5}$, $x_4 = 2$, and $x_1 = 7 - 22x_3 = 7 + \frac{22}{5} = \frac{57}{5}$.

So, the solution is $\mathbf{x} = (\frac{57}{5}, 0, -\frac{1}{5}, 2, 0, 0, 0)$, but it's not a basic feasible solution because one of the entries is negative.

Worksheet 6-3: Q3

Let S be a closed, bounded, and convex set. Figure 1a shows a sampling of points from S that includes all the extreme points and some interior points. Answer the following questions.

- (a) Mark the extreme points $\text{ext}(S)$ on Fig. 1b.
- (b) Highlight the region that represents the convex hull of the extreme points $\text{conv}(\text{ext}(S))$ on Fig. 1c.
- (c) What is your best guess on what the full set S looks like? Explain your answer.

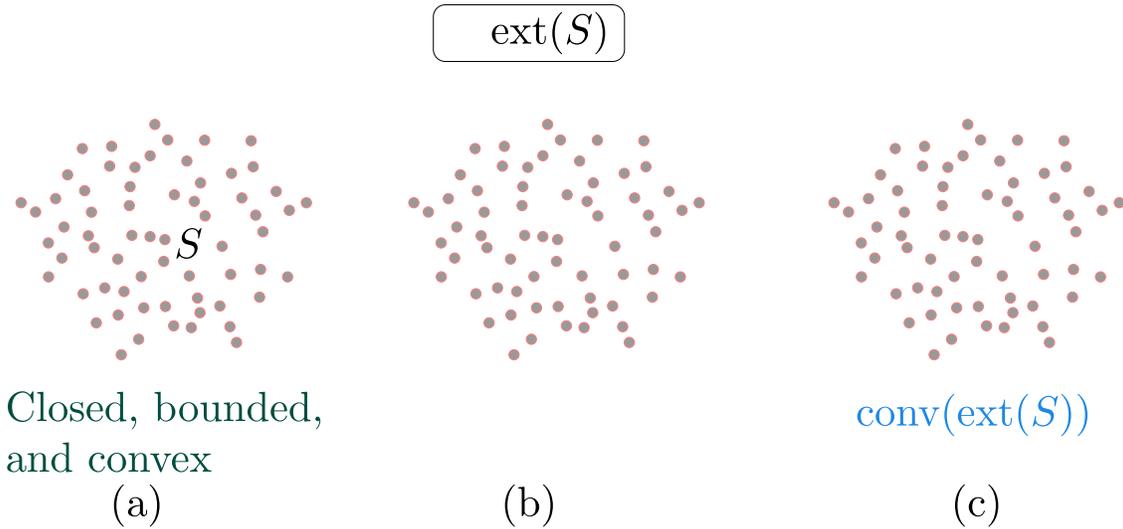


Figure 1: Problem 2.

Answer:

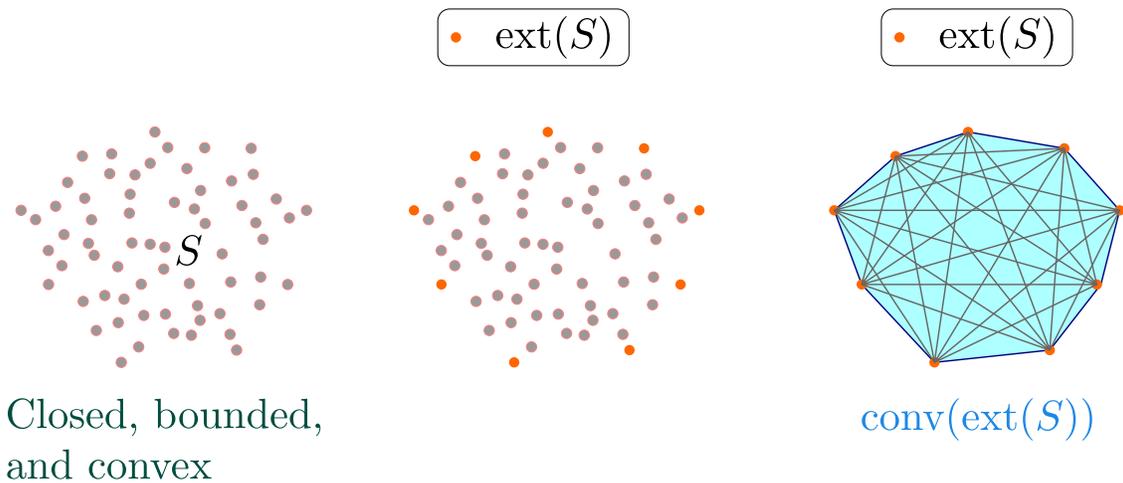


Figure 2: Solution to parts (a) (middle plot), and (b) (right plot).

(d) **Theorem:** Let $S \subseteq \mathbb{R}^n$ be a closed and bounded convex set. Then $S = \text{conv}(\text{ext}(S))$. So S is exactly the convex hull of the extreme points of S shown in Fig. 2c.