Convex Optimization: Part 2

Lecture 8-2 - CMSE 382

Prof. Elizabeth Munch

Michigan State University
::
Dept of Computational Mathematics, Science & Engineering

Mon, Mar 16, 2026

This Lecture

Topics:

- The orthogonal projection operator
- Projection on the non-negative orthant
- Projection on $B[0, r]$

Announcements:

- Homework 4 due Friday

Section 1

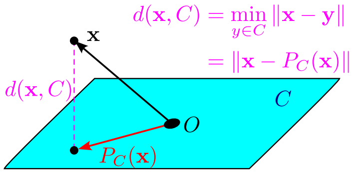
Orthogonal Projection Operator

Orthogonal projection

Definition (Orthogonal projection operator)

Given a nonempty closed convex set C, the orthogonal projection operator $P_C : \mathbb{R}^n \to C$ is defined by

$$P_C(\mathbf{x}) = \arg\min \|\mathbf{y} - \mathbf{x}\|^2 : \mathbf{y} \in C.$$



- Returns the vector in C that is closest to input vector $\mathbf{x}$.
- Is a convex optimization problem:

$$\begin{aligned} \min \quad & \|\mathbf{y} - \mathbf{x}\|^2 \\ \text{s.t.} \quad & \mathbf{y} \in C. \end{aligned}$$

Theorem (first projection theorem)

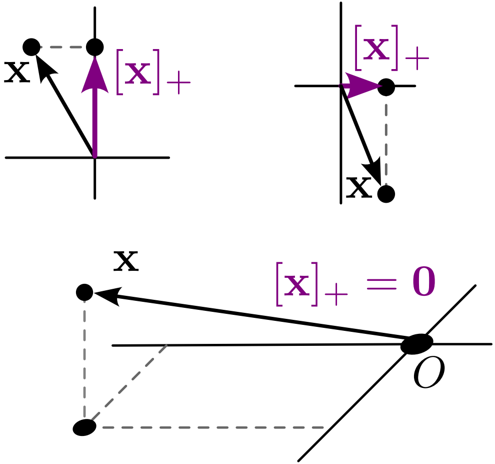
Let C be a nonempty closed convex set. Then the problem $P_C(\mathbf{x}) = \arg\min \|\mathbf{y} - \mathbf{x}\|^2 : \mathbf{y} \in C$ *has a unique optimal solution.*

- Computing $P_C(\mathbf{x})$ can be difficult. Examples where it is easy to compute:
  - Projection on non-negative orthant.
  - Projection onto balls.

Non-negative part of a vector

Definition (Non-negative part of a vector)

- For $\alpha \in \mathbb{R}$, the non-negative part of α is
$$[\alpha]_+ = \begin{cases} \alpha, & \alpha \geq 0 \\ 0, & \alpha < 0. \end{cases}$$
- For a vector $\mathbf{v} \in \mathbb{R}^n$, the non-negative part of $\mathbf{v}$ is $[\mathbf{v}]_+ = \begin{bmatrix} [v_1]_+ \\ [v_2]_+ \\ \vdots \\ [v_n]_+ \end{bmatrix}$



Orthogonal projection: Projection on the non-negative orthant

Let $C = \mathbb{R}^n_+$. To find the orthogonal projection of $\mathbf{x} \in \mathbb{R}^n$ onto $\mathbb{R}^n_+$:

$P_C(\mathbf{x})$

$$\begin{aligned} \min \quad & \|\mathbf{y} - \mathbf{x}\|^2 \\ \text{s.t.} \quad & \mathbf{y} \in C. \end{aligned}$$

Equivalently,

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} (y_i - x_i)^2 \\ \text{s.t.} \quad & y_1, y_2, \ldots, y_n \geq 0. \end{aligned}$$

Separable

$$\begin{aligned} \min \quad & (y_i - x_i)^2 \\ \text{s.t.} \quad & y_i \geq 0. \end{aligned}$$

Definition (Separable convex optimization problems)

A convex optimization problem is called **separable** if the objective function and the constraints can be decomposed into components that each depend on one control/decision variable:

- Objective function: $f(\mathbf{x}) = \sum f_i(\mathbf{x}_i)$.
- Constraint(s): $g(\mathbf{x}) = \sum g_i(\mathbf{x}_i)$, or $\{g_i(\mathbf{x}_i)\}_i$

Orthogonal projection: Projection on the non-negative orthant

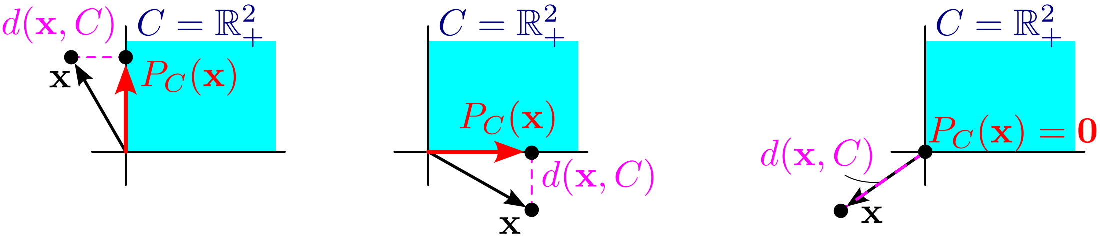
Let $C = \mathbb{R}^n_+$. The orthogonal projection of $\mathbf{x} \in \mathbb{R}^n$ onto $\mathbb{R}^n_+ = \{\mathbf{y} \in \mathbb{R}^n \mid y_i \geq 0 \ \forall i\}$ is

$$\begin{aligned} \min \quad & (y_i - x_i)^2 \\ \text{s.t.} \quad & y_i \geq 0. \end{aligned}$$

Orthogonal projection onto $\mathbb{R}^n_+$

The orthogonal projection operator onto $\mathbb{R}^n_+$ is

$$P_{\mathbb{R}^n_+} = [\mathbf{x}]_+.$$

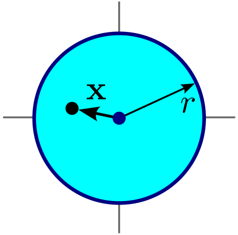


Orthogonal projection: Projection onto balls

Let $C = B[\mathbf{0}, r] = \{\mathbf{y} : \|\mathbf{y} \leq r\}$. The projection of $\mathbf{x}$ onto C is

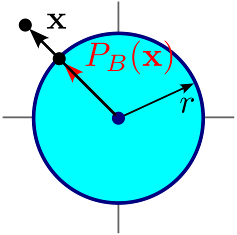
$$\min_{\mathbf{y}} \{\|\mathbf{y} - \mathbf{x}\|^2 : \|\mathbf{y}\|^2 \leq r^2\}$$

$\|\mathbf{x}\| \leq r$



$P_B(\mathbf{x}) = \mathbf{x}$

$\|\mathbf{x}\| > r$



$P_B(\mathbf{x}) \in \text{boundary}(B[0, r])$

Groups - Round 4

Group 1
Michal, Joseph, Saitej, Dev

Group 2
Kyle, Dori, Shanze, Jack

Group 3
Noah, Daniel, Lora, Scott

Group 4
Lowell, Tianjian, Aidan, Anthony

Group 5
Abigail, Breena, Arjun, Luis

Group 6
Purvi, Atticus, Andrew, Vinod

Group 7
Yousif, Jay, Arya, Morgan

Group 8
Jonid, Jake, Dominic, Maye

Group 9
Alice, K M Tausif, Monirul Amin, Ha

Group 10
Jamie, Zheng, Aaron, Long

Group 11
Lauryn, Karen, Sanskaar, Braedon

Group 12
Sai, Brandon, Igor, Quang Minh