Mathematical Preliminaries - Part 2

Lecture 1-2 - CMSE 382

Prof. Elizabeth Munch

Michigan State University
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Dept of Computational Mathematics, Science & Engineering

Weds, Jan 14, 2026

Topics covered

- Eigenvalues and eigenvectors
- Singular values
- Matrix norms

Groups - Round 1

Group 1
Jonid M.
Sanskaar M.
Andrew B.
Zheng Y.

Group 2
Abigail P.
Braedon P.
Vinod R.
Aidan S.

Group 3
Luis C.
Tianjian X.
Lowell M.
Arjun R.

Group 4
Jay B.
Maye B.
Jamie L.
Kyle S.

Group 5
Aaron N.
Purvi G.
Morgan F.
Breena K.

Group 6
Brandon G.
Dominic V.
K M Tausif S.
Anthony K.

Group 7
M.A. Mahin
Dan E.
Quang Minh D.
Ha N.

Group 8
Lora S.
Jack C.
Noah M.
Michal T.

Group 9
Atticus B.
Shanze O.
Joseph M.
Arya S.

Group 10
Scott W.
Karen S.
Dev A.
Dori C.

Group 11
Saitej B.
Jake R.
Alice S.
Long N.

Group 12
Lauryn C.
Sai P.
Yousif E.

Eigenvalues and eigenvectors

Definition

Given a square matrix $A \in \mathbb{R}^{n \times n}$, a scalar $\lambda \in \mathbb{R}$ is an **eigenvalue** of A if there exists a non-zero vector $v \in \mathbb{R}^n$ such that

$$Av = \lambda v.$$

The vector v is called an **eigenvector** corresponding to the eigenvalue λ.

Solving for eigenvalues and eigenvectors:

1. Rewrite as $(A - \lambda I)v = 0$.
2. For non-trivial solutions: $\det(A - \lambda I) = 0$.
3. Solve for λ (characteristic polynomial).
4. Plug λ back in to solve for v.

Other values:

- The **spectrum** of A is the set of all eigenvalues of A.
- $\text{trace}(A) = \sum_{i=1}^{n} \lambda_i$.
- $\det(A) = \prod_{i=1}^{n} \lambda_i$.

Singular values

Definition (Singular values)

Given a real matrix $A \in \mathbb{R}^{m \times n}$, the **singular values** of A are defined as the square roots of the eigenvalues of $A^T A$.

Properties of singular values:

- If σ is a singular value of A, then there exists a non-zero vector $v \in \mathbb{R}^n$ such that

$$A^T A v = \sigma^2 v.$$

- All singular values are non-negative, i.e., $\sigma \geq 0$.
- The number of non-zero singular values is equal to the rank of A.
- Singular values provide information about the scaling and stretching properties of the linear transformation represented by A.
- For a symmetric real matrix ($A = A^T$), the singular values are the absolute values of the eigenvalues.

Norms of matrices

Definition

A **norm** on the set of real matrices $\mathbb{R}^{m\times n}$ is a function $\|\cdot\| : \mathbb{R}^{m\times n} \to \mathbb{R}$ that satisfies for all $\alpha \in \mathbb{R}$ and $A, B \in \mathbb{R}^{m\times n}$ the following:

- $\|A\| \geq 0$ (positive-valued)
- $\|A\| = 0$ if and only if $A = \mathbf{0}_{m,n}$
- $\|\alpha A\| = |\alpha|\|A\|$ (positive homogeneity)
- $\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality)

Induced matrix norm

Definition (Induced matrix norm)

Given a real matrix $A \in \mathbb{R}^{m\times n}$, as well as the norms $\|\cdot\|_a$ and $\|\cdot\|_b$ defined on $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Define the **induced matrix norm**

$$\|A\|_{a,b} = \max_{\|x\|_a=1} \|Ax\|_b.$$

1. For L_p norms, if $a = b = p$, we write $\|A\|_p$ instead of $\|A\|_{p,p}$.
2. This norm returns the maximum of all linear transformations (measured using b) of input vectors of unit length (as measured by a).
3. It measures the maximum 'stretching' of unit vectors (as measured by a) in $\mathbb{R}^n$ due the action of A

Examples for p-norms

Definition

The **matrix p-norm** for $A \in \mathbb{R}^{m \times n}$ is induced by the corresponding p-norm for vectors in $\mathbb{R}^n$ and $\mathbb{R}^m$:

$$\|A\|_p = \max_{\|x\|_p = 1} \|Ax\|_p.$$

Examples:

- $p = 1$: max absolute *column* sum,
  $$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{m} |a_{ij}|.$$
- $p = 2$: largest singular value of A, AKA square root of largest eigenvalue of $A^T A$:
  $$\|A\|_2 = \sigma_{\max}(A) = \sqrt{\lambda_{\max}(A^T A)}.$$
  - **Spectral norm**, default assumption for $\|A\|$ if no subscript is given.
- $p = \infty$: max absolute *row* sum,
  $$\|A\|_\infty = \max_{1 \le i \le m} \sum_{j=1}^{n} |a_{ij}|.$$

Definition (Frobenius norm)

The **Frobenius norm** for $A \in \mathbb{R}^{m \times n}$ is defined as

$$\|A\|_F = \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2}.$$

Other versions:

- $\|A\|_F = \sqrt{\text{trace}(A^T A)}$
- $\|A\|_F = \sqrt{\sum_{i=1}^{\min\{m,n\}} \sigma_i^2}$, where σ_i are the singular values of A.

Definition (Nuclear norm)

The **nuclear norm** (also called **trace norm**) for $A \in \mathbb{R}^{m\times n}$ is defined as

$$\|A\|_* = \sum_{i=1}^{\min\{m,n\}} \sigma_i,$$

where σ_i are the singular values of A.

Other versions:

- $\|A\|_* = \text{trace}(\sqrt{A^T A})$ where $\sqrt{A^T A} = B$ is the positive semi-definite square root of $A^T A$, so that $BB = A^T A$.

Group Work Time

- Introduce yourself to your group members
- Find and download the worksheet from the course website.
- In the last 10 minutes of class, you will upload your completed worksheet to D2L. Note: graded on completion only.

Next time

- Watch videos posted on the course webpage
- Check syllabus dates - Especially exam, homework, and quiz dates
- Office hours