Name: **Present group members:**

Worksheet 5-1: Q1

Find the next iterate of $f(x, y) = 100x^4 + 0.01y^4$ with initial vector $\mathbf{x}_0 = (1, 1)^\top$ using pure Newton's method by following the following steps.

1. Calculate the gradient $\nabla f(\mathbf{x}_0)$ and the Hessian $\nabla^2 f(\mathbf{x}_0)$.

   - $\nabla f(x, y) = \begin{bmatrix} 400x^3 \\ 0.04y^3 \end{bmatrix}$
   - $\nabla f(\mathbf{x}_0) = \begin{bmatrix} 400 \\ 0.04 \end{bmatrix}$
   - $\nabla^2 f(x, y) = \begin{bmatrix} 1200x^2 & 0 \\ 0 & 0.12y^2 \end{bmatrix}$
   - $\nabla^2 f(\mathbf{x}_0) = \begin{bmatrix} 1200 & 0 \\ 0 & 0.12 \end{bmatrix}$

2. What is the Newton direction $\mathbf{d}_0$?

   - $\mathbf{d}_k$ is the solution to $\nabla^2 f(\mathbf{x}_k)\mathbf{d}_k = -\nabla f(\mathbf{x}_k)$, which can be rewritten as

     $$\mathbf{d}_k = -(\nabla^2 f(\mathbf{x}_k))^{-1} \nabla f(\mathbf{x}_k)$$

     where we plug in $\nabla f(\mathbf{x}_0)$ and $\nabla^2 f(\mathbf{x}_0)$ from part (a).
   - Here that is

     $$\mathbf{d}_0 = -\begin{bmatrix} 1200 & 0 \\ 0 & 0.12 \end{bmatrix}^{-1} \begin{bmatrix} 400 \\ 0.04 \end{bmatrix} = -\begin{bmatrix} \frac{1}{1200} & 0 \\ 0 & \frac{1}{0.12} \end{bmatrix} \begin{bmatrix} 400 \\ 0.04 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$$

   - So $\mathbf{d}_0 = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix}$

3. What is $\mathbf{x}_1$?

   - $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$
   - $\mathbf{x}_1 = \mathbf{x}_0 + \mathbf{d}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}$

Worksheet 5-1: Q2

Consider the function $f(x) = \sqrt{1+x^2}$ defined over $\mathbb{R}$. The first and second derivatives are given by $f'(x) = \frac{x}{\sqrt{1+x^2}}$ and $f''(x) = \frac{1}{(1+x^2)^{3/2}}$, respectively.

1. Find the Newton direction d_k in terms of x_k.
   - d_k is the solution to $f''(x_k)d_k = -f'(x_k)$, which can be rewritten as

     $$d_k = -\frac{f'(x_k)}{f''(x_k)}$$

     where we plug in $f'(x_k)$ and $f''(x_k)$ from above.
   - Here that is

     $$d_k = -\frac{\frac{x_k}{\sqrt{1+x_k^2}}}{\frac{1}{(1+x_k^2)^{3/2}}} = -x_k(1+x_k^2)$$
   - So $d_k = -x_k(1+x_k^2)$.

2. Use pure Newton method to derive an expression for x_{k+1} in terms of x_k.
   - $x_{k+1} = x_k + d_k$
   - $x_{k+1} = x_k - x_k(1+x_k^2) = -x_k^3$

3. What choices of x_0 will lead to convergence to $x^* = 0$?
   - We want to find x_0 such that the sequence defined by $x_{k+1} = -x_k^3$ converges to 0.
   - If $|x_0| < 1$, then $|x_k|$ will get smaller and smaller as k increases, so the sequence will converge to 0.
   - If $|x_0| > 1$, then $|x_k|$ will get larger and larger as k increases, so the sequence will diverge.
   - If $|x_0| = 1$, then x_k will alternate between 1 and -1, so the sequence will not converge to 0.
   - So the choices of x_0 that will lead to convergence to $x^* = 0$ are those such that $|x_0| < 1$.
   - Note that we can't actually use the theorem for this example because $f''(0) = 0$ so there's no positive $m > 0$ that would make $f''(x) \geq m$. However, even though it doesn't satisfy the theorem, it happens to converge anyway.

When you're done with the worksheet, go download the Jupyter notebook to try coding this.