Name: **Present group members:**

Worksheet 2-2: Q1

For the following symmetric matrices, cross out the classifications it **CANNOT** be due to the diagonal entries (don't calculate the eigenvalues).

- $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix}$ *pos def* *pos semidef* *indefinite* *neg def* *neg semidef*

  Can't be negative definite or negative semidefinite because diagonal entry of 5 is positive. Also can't be positive definite because diagonal entry of 0 is not positive. Remaining options are positive semidefinite or indefinite.

- $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$ *pos def* *pos semidef* *indefinite* *neg def* *neg semidef*

  Can't be negative definite or negative semidefinite because diagonal entries are positive. Remaining options are positive definite, positive semidefinite, or indefinite.

- $A_3 = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$ *pos def* *pos semidef* *indefinite* *neg def* *neg semidef*

  Only option is indefinite because one diagonal entry is positive and one is negative.

- $A_4 = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix}$ *pos def* *pos semidef* *indefinite* *neg def* *neg semidef*

  Can't be positive definite or positive semidefinite because diagonal entries are negative. Remaining options are negative definite, negative semidefinite, or indefinite.

For the following matrices with their given eigenvalues, what is the correct classification and why?

- $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 5 \end{bmatrix}$; eigvals $= \{-0.193, 5.193\}$

  pos def *pos semidef* *indefinite* *neg def* *neg semidef*

  Indefinite because one positive and one negative eigenvalue

- $A_2 = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$; eigvals $= \{0.382, 2.618\}$

  pos def *pos semidef* *indefinite* *neg def* *neg semidef*

  Positive definite because all eigenvalues are positive

- $A_3 = \begin{bmatrix} -1 & 2 \\ 2 & 1 \end{bmatrix}$; eigvals $= \{-2.236, 2.236\}$

  pos def *pos semidef* *indefinite* *neg def* *neg semidef*

  Indefinite because one positive and one negative eigenvalue

- $A_4 = \begin{bmatrix} -3 & 1 \\ 6 & -2 \end{bmatrix}$; eigvals $= \{-5, 0\}$

  pos def *pos semidef* *indefinite* *neg def* *neg semidef*

  Negative semidefinite because eigenvalues are either negative or 0.

Worksheet 2-2: Q2 On the last worksheet, we found the stationary points of the function

$$f(x, y) = 6x^2y - 3x^3 + 2y^3 - 150y$$

were at $p_1 = (0, 5)$ $p_2 = (0, -5)$ $p_3 = (4, 3)$ $p_4 = (-4, -3)$. Use the second optimality condition to classify each stationary point as a local minimum, local maximum, or saddle point.

You may use a computational tool to compute eigenvalues for you (although in these examples you shouldn't have to!). You can visually check your answer at desmos.com/3d/xa4komuwmb.

- $\nabla f(x) = \begin{bmatrix} 12xy - 9x^2 \\ 6x^2 + 6y^2 - 150 \end{bmatrix}$
- $\nabla^2 f(x) = \begin{bmatrix} 12y - 18x & 12x \\ 12x & 12y \end{bmatrix}$
- At $p_1 = (0, 5)$, $\nabla^2 f(p_1) = \begin{bmatrix} 60 & 0 \\ 0 & 60 \end{bmatrix}$. This has eigenvalues 60 and 60 (yay for diagonal matrix!) which are both positive, so it is positive definite $\implies$ local minimum.
- At $p_2 = (0, -5)$, $\nabla^2 f(p_2) = \begin{bmatrix} -60 & 0 \\ 0 & -60 \end{bmatrix}$.

  This has eigenvalues -60 and -60 (again easy because it's a diagonal matrix) which are both negative, so it is negative definite $\implies$ local maximum.
- At $p_3 = (4, 3)$, $\nabla^2 f(p_3) = \begin{bmatrix} -36 & 48 \\ 48 & 36 \end{bmatrix}$.

  This has eigenvalues 60 and -60 (using a computational tool) which have opposite signs, so it is indefinite $\implies$ saddle point.

  Note, can also see that it is indefinite because it has one positive and one negative diagonal entry.
- At $p_4 = (-4, -3)$, $\nabla^2 f(p_4) = \begin{bmatrix} 36 & -48 \\ -48 & -36 \end{bmatrix}$.

  This also has eigenvalues 60 and -60 (using a computational tool) which have opposite signs, so it is indefinite $\implies$ saddle point.

  Note, can also see that it is indefinite because it has one positive and one negative diagonal entry.

Worksheet 2-2: Q3

Find and classify the stationary points of $f(x) = 2x^3 + 3y^2 + 3x^2y - 24y$

- $\nabla f(x, y) = \begin{bmatrix} 6x^2 + 6xy \\ 6y + 3x^2 - 24 \end{bmatrix}$
- Set gradient to 0 to find stationary points
  - From the first equation $6x^2 + 6xy = 6x(x + y) = 0$ means either $x = 0$ or $y = -x$
  - From the second equation $6y + 3x^2 - 24 = 0 \implies y = -\frac{1}{2}x^2 + 4$
- If $x = 0$, then from second equation $y = 4$ so one stationary point is at $(0, 4)$
- If $y = -x$, then substituting into second equation gives

$$\begin{aligned} 6(-x) + 3x^2 - 24 &= 0 \\ 3x^2 - 6x - 24 &= 0 \\ x^2 - 2x - 8 &= 0 \\ (x - 4)(x + 2) &= 0 \end{aligned}$$

  So $x = 4$ or $x = -2$.

  - If $x = 4$, then $y = -4$ so another stationary point is at $(4, -4)$
  - If $x = -2$, then $y = 2$ so another stationary point is at $(-2, 2)$
- So the stationary points are at $(0, 4)$, $(4, -4)$, and $(-2, 2)$
- Next, compute the Hessian

$$\nabla^2 f(x, y) = \begin{bmatrix} 12x + 6y & 6x \\ 6x & 6 \end{bmatrix}$$

- Evaluate Hessian at each stationary point and classify
  - At $(0, 4)$, $\nabla^2 f(0, 4) = \begin{bmatrix} 24 & 0 \\ 0 & 6 \end{bmatrix}$ This has eigenvalues 24 and 6 (yay for diagonal matrix!) which are both positive, so it is positive definite $\implies$ local minimum.
  - At $(4, -4)$, $\nabla^2 f(4, -4) = \begin{bmatrix} 24 & 24 \\ 24 & 6 \end{bmatrix}$ This has eigenvalues 40.63 and -10.63 (using a computational tool) which have opposite signs, so it is indefinite $\implies$ saddle point.
  - At $(-2, 2)$, $\nabla^2 f(-2, 2) = \begin{bmatrix} -12 & -12 \\ -12 & 6 \end{bmatrix}$ This has eigenvalues -18 and 12 (using a computational tool or by hand) which have opposite signs, so it is indefinite $\implies$ saddle point.
- Visual check: desmos.com/3d/juzovyr3hd