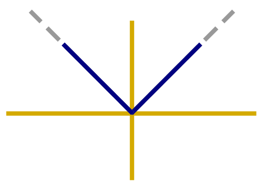
Name: **Present group members:**

Worksheet 6-2: Q1

Indicate whether each of the given sets is a cone. If the set is a cone, indicate whether it is a convex cone or not. You must justify your answers.

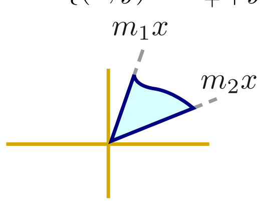
(a) $S = \{(x, |x|) \mid x \in \mathbb{R}\}$



For any positve $\lambda \geq 0$, the point $(\lambda x, |\lambda x|) = \lambda(x, |x|)$ is in the set, so this is a cone.

We have a theorem that a cone is convex if and only if $\mathbf{x} + \mathbf{y}$ is in the set for any $\mathbf{x}, \mathbf{y}$ in the set. Consider the points $(-1, 1)$ and $(1, 1)$, which are both in the set. But their sum is $(0, 2)$, which is not in the set. So this is not a convex cone.

(b) Fix $m_1, m_2 \in \mathbb{R}$ that are not equal to each other, $m_1 \neq m_2$.
$S = \{(x, y) \in \mathbb{R}^2_+ \mid y \leq m_1 x \text{ and } y \geq m_2 x\}$



For any $\lambda \geq 0$, if (x, y) is in the set we know that $y \leq m_1 x$ and $y \geq m_2 x$ for some $m_1, m_2 \in \mathbb{R}$.

Then $\lambda(x, y) = (\lambda x, \lambda y)$, and multiplying the inequalities above by λ (allowed because $\lambda \geq 0$), $\lambda y \leq m_1 \lambda x$ and $\lambda y \geq m_2 \lambda x$. So $(\lambda x, \lambda y)$ is in the set when (x, y) in the set. So this is a cone.

For convex cone, pick two points (x_1, y_1) and (x_2, y_2) in the set. So

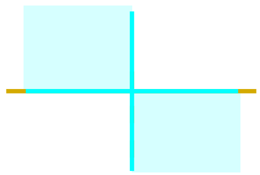
$$y_1 \leq m_1 x_1 \text{ and } y_1 \geq m_2 x_1$$
$$y_2 \leq m_1 x_2 \text{ and } y_2 \geq m_2 x_2$$

Then add the equations above to get

$$y_1 + y_2 \leq m_1 x_1 + m_1 x_2 \Rightarrow y_1 + y_2 \leq m_1(x_1 + x_2)$$
$$y_1 + y_2 \geq m_2 x_1 + m_2 x_2 \Rightarrow y_1 + y_2 \geq m_2(x_1 + x_2)$$

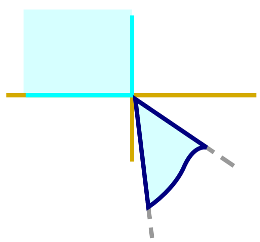
This means $(x_1 + x_2, y_1 + y_2)$ is in the set. So S is a convex cone.

(c) $S = \{(x,y) \mid x \leq 0, y \geq 0\} \cup \{(x,y) \mid x \geq 0, y \leq 0\}$



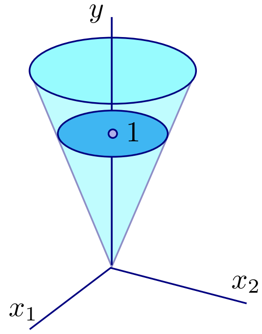
- *Each quadrant is a cone: for any $\lambda \geq 0$, if (x,y) is in the first quadrant, then $\lambda(x,y)$ is in the first quadrant. Similarly for the third quadrant. So each quadrant is a cone.*
- *The union of cones is a cone, so S is a cone.*
- *However, S is not a convex cone. Consider the points $(0,1)$ and $(1,0)$, which are both in the set. But their sum is $(1,1)$, which is not in the set. So this is not a convex cone.*

(d) The shown region (curved boundary on the bottom means it goes on forever in that direction).



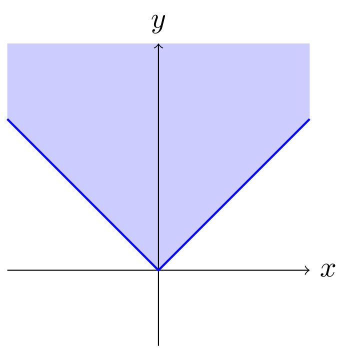
- *This is the union of two cones (one is a quadrant, the other is of the form of part (b)), so this is a cone.*
- *Again, not a convex cone. Pick a point on the y-axis like $(0,1)$ and the other on the top blue line of the bottom cone. Then the line between them is not in the set. So this is not a convex cone.*

(e) Lorentz cone: $\{(x_1, x_2, y) \in \mathbb{R}^3 \mid \|(x_1, x_2)\| \leq y\}$



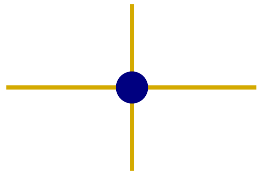
- *For any $\lambda \geq 0$, if (x_1, x_2, t) is in the set, then $\lambda(x_1, x_2, t) = (\lambda x_1, \lambda x_2, \lambda t)$ is in the set because $\|(\lambda x_1, \lambda x_2)\| \leq |\lambda| \|(x_1, x_2)\|$ (allowed to multiply the inequality by λ because $\lambda \geq 0$). So this is a cone.*
- *For convex cone, pick two points (a_1, b_1, t_1) and (a_2, b_2, t_2) in the set.*
- *So $\|(a_1, b_1)\| \leq t_1$ and $\|(a_2, b_2)\| \leq t_2$.*
- *Add the equations above and use the triangle inequality from the definition of norms to get*
$$\|(a_1 + a_2, b_1 + b_2)\| \leq \|(a_1, b_1)\| + \|(a_2, b_2)\| \leq t_1 + t_2.$$
- *So $(a_1 + a_2, b_1 + b_2, t_1 + t_2)$ is in the set, which means S is a convex cone.*

(f) $S = \{(x, y) \in \mathbb{R}^2 \mid y \geq |x|\}$.



- *For any $\lambda \geq 0$, if (x, y) is in the set, then $\lambda(x, y) = (\lambda x, \lambda y)$ is in the set because $\lambda y \geq |\lambda x|$ (allowed to multiply the inequality by λ because $\lambda \geq 0$). So this is a cone.*
- *For convex cone, pick two points (x_1, y_1) and (x_2, y_2) in the set. So $y_1 \geq |x_1|$ and $y_2 \geq |x_2|$. Then add the equations above to get $y_1 + y_2 \geq |x_1| + |x_2| \geq |x_1 + x_2|$ (by triangle inequality). So $(x_1 + x_2, y_1 + y_2)$ is in the set. So S is a convex cone.*

(g) $\{\mathbf{0} \in \mathbb{R}^n\}$



- *The only point in the set is* 0*, and* $\lambda \cdot 0 = 0$*, so this is (technically) a cone.*
- *Same trick, for any* $x, y \in S$ *(where now* x *and* y *both have to be 0),* $x + y = 0$ *so it is also in the set. So this is a convex cone.*