Mathematical Preliminaries - Part 3

Lecture 1-3 - CMSE 382

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Topics covered

- Topological concepts
  - Open and closed sets
  - Interior and boundary points
  - Level sets
  - Bounded and compact sets
- Differentiability
  - Hessian
  - directional derivative

Groups - Round 1

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Tianjian X.
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Purvi G.
Morgan F.
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Group 6
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Dominic V.
K M Tausif S.
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Group 7
M.A. Mahin
Dan E.
Quang Minh D.
Ha N.

Group 8
Lora S.
Jack C.
Noah M.
Michal T.

Group 9
Atticus B.
Shanze O.
Joseph M.
Arya S.

Group 10
Scott W.
Karen S.
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Group 11
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Group 12
Lauryn C.
Sai P.
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Section 1

Topological concepts

Open and closed balls

Definition

For a choice of norm $\|\cdot\|$ on $\mathbb{R}^n$, the **open ball** of radius $r > 0$ centered at $\mathbf{c} \in \mathbb{R}^n$ is defined as

$$B(\mathbf{c}, r) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{c}\| < r\}.$$

The **closed ball** of radius $r > 0$ centered at $\mathbf{c} \in \mathbb{R}^n$ is defined as

$$B[\mathbf{c}, r] := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{c}\| \leq r\}.$$

Note: Assume Euclidean norm unless told otherwise.

Important subsets of $\mathbb{R}^n$

- **Nonnegative orthant**:
  $\mathbb{R}^n_+ = \{\mathbf{x} \in \mathbb{R}^n : x_i \geq 0, i = 1, \ldots, n\}$
- **Positive orthant**:
  $\mathbb{R}^n_{++} = \{\mathbf{x} \in \mathbb{R}^n : x_i > 0, i = 1, \ldots, n\}$
- **Closed line segment:**
  $[\mathbf{x}, \mathbf{y}] = \{\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} : \lambda \in [0, 1]\}$
- **Open line segment:**
  $(\mathbf{x}, \mathbf{y}) = \{\lambda \mathbf{x} + (1 - \lambda)\mathbf{y} : \lambda \in (0, 1)\}$
- **Unit-simplex**:
  $\Delta_n = \{\mathbf{x} \in \mathbb{R}^n_+ : \sum_{i=1}^n x_i = 1\}$

Interior points

Definition

Given a set $U \subseteq \mathbb{R}^n$, a point $\mathbf{c} \in U$ is an **interior point** of U if there exists an open ball $B(\mathbf{c}, r)$ for some $r > 0$ such that $B(\mathbf{c}, r) \subseteq U$.

The set of all interior points of U is called the **interior** of U and is denoted by $\text{int}(U)$.

Examples:

- $\text{int}(B[c, r]) = B(c, r)$
- $\text{int}(\mathbb{R}^n_+) = \mathbb{R}^n_{++}$

Open sets

Definition

A set $U \subseteq \mathbb{R}^n$ is **open** if every point in U is an interior point of U, i.e., for every $\mathbf{c} \in U$, there exists an open ball $B(\mathbf{c}, r)$ such that $B(\mathbf{c}, r) \subseteq U$.

Examples:

- $B(c, r)$ is open
- $B[c, r]$ is not open
- $\mathbb{R}^n$ is open
- Union of open sets is open
- Finite intersection of open sets is open

Boundary points

Definition

Given a set $U \subseteq \mathbb{R}^n$, a point $\mathbf{c} \in \mathbb{R}^n$ is a **boundary point** of U if for every $r > 0$, the open ball $B(\mathbf{c}, r)$ contains at least one point in U and at least one point not in U.

Closed set

Definition

A set $U \subseteq \mathbb{R}^n$ is **closed** if

- it contains all the limits of convergent sequences
- its complement $U^c = \mathbb{R}^n \setminus U = \{x \in \mathbb{R}^n \mid x \notin U\}$ is open
- it contains all its boundary points

Examples:

- $B[c, r]$ is closed
- Closed line segment $[\mathbf{x}, \mathbf{y}]$ is closed
- $\mathbb{R}^n_+$ is closed
- The unit simplex Δ_n is closed

Levelsets

Proposition

Let f be a continuous function defined over a closed set $S \subseteq \mathbb{R}^n$. Then for any $\alpha \in \mathbb{R}$, the sublevel set and contour sets

- $Lev(f, \alpha) = \{\mathbf{x} \in S : f(\mathbf{x}) \leq \alpha\}$, *and*
- $Con(f, \alpha) = \{\mathbf{x} \in S : f(\mathbf{x}) = \alpha\}$,

are closed.

desmos.com/3d/rdxctxv0uo

Bounded and compact

Definition

A set $U \subseteq \mathbb{R}^n$ is **bounded** if there exists a real number $M > 0$ such that $\|\mathbf{x}\| \leq M$ for all $\mathbf{x} \in U$.
A set $U \subseteq \mathbb{R}^n$ is **compact** if it is closed and bounded.

Section 2

Differentiability

Definition

The **gradient** of a scalar-valued function $f : \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{x}$ is defined as

$$\nabla f(\mathbf{x}) := \begin{bmatrix} \frac{\partial f}{\partial x_1}(\mathbf{x}) \\ \frac{\partial f}{\partial x_2}(\mathbf{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\mathbf{x}) \end{bmatrix}.$$

The operator ∇ is read 'nabla' or 'del,' and $\frac{\partial f}{\partial x_i}$ is the ith **partial derivative** of f at $\mathbf{x}$.

- A function f is **continuously differentiable** over an open set U if the gradient exists and is continuous on U.
- A function f is **twice continuously differentiable** over an open set U if the gradient is continuously differentiable on U; or if you can take all second partial derivatives and they are continuous on U.

Directional Derivative

Scalar-valued function

Definition

The **directional derivative** of $f : \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{x}$ along the direction $\mathbf{d}$ is defined as $f'(\mathbf{x}; \mathbf{d}) = \nabla f(\mathbf{x})^\top \mathbf{d}$.

- Gives the instantaneous rate of change of f along direction $\mathbf{v}$ through point $\mathbf{x}$.

desmos.com/3d/ojt8rjazr7

Hessian

Scalar-valued function

Definition

The **Hessian** of a scalar-valued function $f : \mathbb{R}^n \to \mathbb{R}$ at $\mathbf{x}$ is defined as the $n \times n$ symmetric matrix

$$\nabla^2 f(\boldsymbol{x}) = H_{\mathbf{f}}(\boldsymbol{x}) := \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \ddots & \vdots & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}_{\mathbf{x}}$$

- The order of the partial derivatives does not matter: $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$

Hessian

Example: The Hessian of $\mathbf{f}(x, y) = x + 2xy - y^2 + 3$ at the point $(1, 1)$.

- We can calculate the gradient first:
- $\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x}(\boldsymbol{x}) \\ \frac{\partial f}{\partial y}(\boldsymbol{x}) \end{bmatrix} = \begin{bmatrix} 1 + 2y \\ 2x - 2y \end{bmatrix}_{\mathbf{x}}$
- Taking the derivatives of these entries again gives the Hessian:

$$\nabla^2 f(\boldsymbol{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}_{\mathbf{x}} = \begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}_{\mathbf{x}}$$

- So the Hessian at $(1, 1)$ (or for any plugged in point since everything is a constant!) is $\begin{bmatrix} 0 & 2 \\ 2 & -2 \end{bmatrix}$

Linear approximation theorem

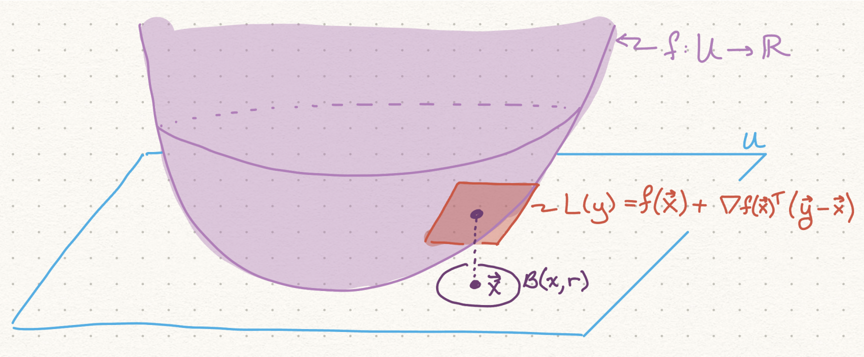
Theorem (Linear Approximation Theorem)

- *Let* $f : U \to \mathbb{R}$ *be a twice continuously differentiable function over an open set* $U \subseteq \mathbb{R}^n$*,*
- *Let* $\mathbf{x} \in U$, $r > 0$ *satisfy* $B(\mathbf{x}, r) \subseteq U$*.*

Then for any $\mathbf{y} \in B(\mathbf{x}, r)$ *there exists* $\boldsymbol{\xi} \in [\mathbf{x}, \mathbf{y}]$ *such that*

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) + \frac{1}{2}(\mathbf{y} - \mathbf{x})^T \nabla^2 f(\boldsymbol{\xi})(\mathbf{y} - \mathbf{x}).$$

Linear approximation visual



$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y}-\mathbf{x}) + \frac{1}{2}(\mathbf{y}-\mathbf{x})^T\nabla^2 f(\xi)(\mathbf{y}-\mathbf{x}).$$

Quadratic approximation theorem

Theorem (Quadratic Approximation Theorem)

- *Let* $f : U \to \mathbb{R}$ *be a twice continuously differentiable function over an open set* $U \subseteq \mathbb{R}^n$.
- *Let* $\mathbf{x} \in U$, $r > 0$ *satisfy* $B(\mathbf{x}, r) \subseteq U$.

Then for any $\mathbf{y} \in B(\mathbf{x}, r)$

$$f(\mathbf{y}) = f(\mathbf{x}) + \nabla f(\mathbf{x})^T(\mathbf{y} - \mathbf{x}) + \frac{1}{2}(\mathbf{y} - \mathbf{x})^T \nabla^2 f(\mathbf{x})(\mathbf{y} - \mathbf{x}) + o(\|\mathbf{y} - \mathbf{x}\|^2).$$

Group Work Time

- Find and download the worksheet from the course website.
- In the last 10 minutes of class, you will upload your completed worksheet to D2L. Note: graded on completion only.

Next time

- Check course webpage for videos and reading for next class
- Office hours will be scheduled for next week

- Quiz 1 on Wednesday, Jan 21.
- Cheat sheet allowed.
  - One 8.5 x 11 inch sheet of paper, front and back.
  - Handwritten only.
  - Cannot be duplicate/photocopy/etc of someone else's work.
  - You will turn this in with your quiz.
  - Failure to follow these rules will result in a 10% deduction on the quiz.
- Calculator allowed (no internet)
- Note: Lowest grade dropped at end of semester.