Name: **Present group members:**

Worksheet 1-2: Q1

For the matrix $A = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$, find

1. The induced 1-norm, $\|A\|_1$.
    - This is max column sum, don't forget absolute values!
    - $\|A\vec{u}\|_1 = \max_j \sum_i |A_{ij}| = \max(4, 6) = 6$

2. The induced ∞-norm, $\|A\|_\infty$.
    - This is max row sum, don't forget absolute values!
    - $\|A\|_\infty = \max_i \sum_j |A_{ij}| = \max(3, 7) = 7$

3. The Frobenius norm, $\|A\|_F$.
    - This is entrywise 2-norm, so square everything and add it
    - $\|A\|_F = \sqrt{\sum_{ij} A_{ij}^2} = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$

Worksheet 1-2: Q2

We are going to determine the induced 2-norm for the matrix $A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}$.

1. Compute $A^T A$.

   $A^T A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1+4 & -2-2 \\ -2-2 & 4+1 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$

2. Find the eigenvalues of $A^T A$. We generally sort these in decreasing order so that $\lambda_1 \geq \lambda_2$.

   - $\det(A^T A - \lambda I) = \det \begin{bmatrix} 5-\lambda & -4 \\ -4 & 5-\lambda \end{bmatrix} = (5-\lambda)^2 - 16 = \lambda^2 - 10\lambda + 9 = (\lambda - 9)(\lambda - 1)$
   - So the eigenvalues are $\lambda_1 = 9$ and $\lambda_2 = 1$.

3. Find the singular values of A.

   - The singular values are the square roots of the eigenvalues above
   - $\sigma_1 = \sqrt{\lambda_1} = 3$
   - $\sigma_2 = \sqrt{\lambda_2} = 1$

4. Compute the induced 2-norm of A.

   - The induced 2-norm is the largest signular value of A, in this case $\|A\|_2 = 3$.

Worksheet 1-2: Q3

1. Show that for $A \in \mathbb{R}^{m\times n}$, if λ is a non-zero eigenvalue of $A^T A$ with eigenvector v, then λ is a non-zero eigenvalue of AA^T with eigenvector Av.

   - If $\lambda \neq 0$ with eigenvector v, then $A^T Av = \lambda v$
   - Multiply both sides by A and add some parentheses:

     $$AA^T(Av) = \lambda(Av)$$

     which is exactly what is needed to shwo that λ is an eigenvalue of AA^T with eigenvector Av.

2. Use the previous statement to show that for $A \in \mathbb{R}^{m\times n}$, $\|A\| = \|A^T\|$ for the spectral norm (AKA induced 2-norm).

   - By the previous question, the non-zero eigenvalues of $A^T A$ and AA^T are the same.
   - Thus, the singular values of A and A^T are the same since they are the square roots of the eigenvalues.
   - Therefore, the induced 2-norms are the same: $\|A\|_2 = \|A^T\|_2$ since they are each the largest singular value.

3. Show that $\|A\|_F^2 = \sum_{i=1}^n \lambda_i(A^T A)$. *(Hint: Consider the trace $A^T A$.)*

   - Entry (j, j) for $A^T A$ is $\sum_{i=1}^m a_{ij}^2$. So

     $$\|A\|_F^2 = \sum_{i=1}^m \sum_{j=1}^n a_{ij}^2 = \sum_{j=1}^n \sum_{i=1}^m a_{ij}^2 = \sum_{j=1}^n (A^T A)_{jj} = \mathrm{trace}(A^T A)$$

     This means (we have this equation on the slides too) $\|A\|_F^2 = \mathrm{trace}(A^T A)$.
   - Since the trace of a matrix is the sum of its eigenvalues, we have $\|A\|_F^2 = \sum_{i=1}^n \lambda_i(A^T A)$.