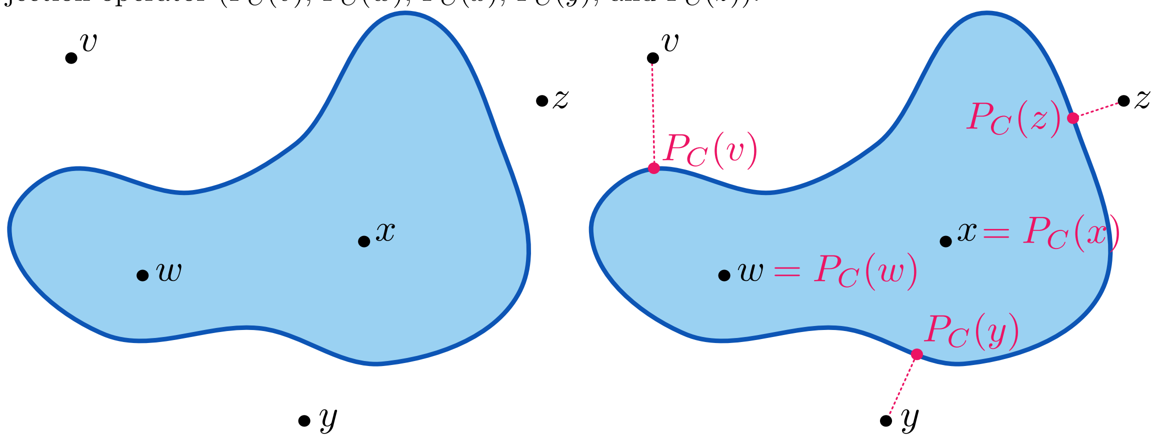
Name: **Present group members:**

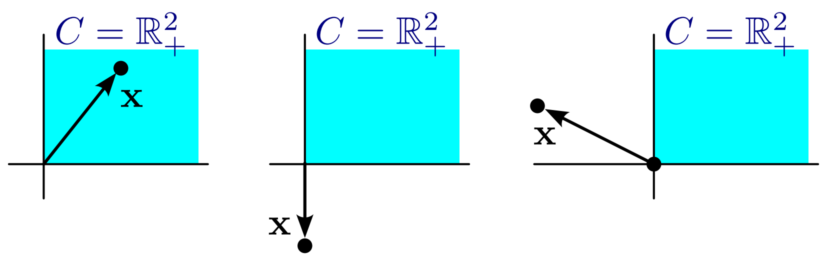
Worksheet 8-2: Q1

For the following sets and each point drawn (v, w, x, y, and z), mark the point that minimizes the projection operator ($P_C(v)$, $P_C(w)$, $P_C(x)$, $P_C(y)$, and $P_C(z)$).



Worksheet 8-2: Q2

For each of the shown vectors $\mathbf{x}$, answer the following



(a) (b) (c)

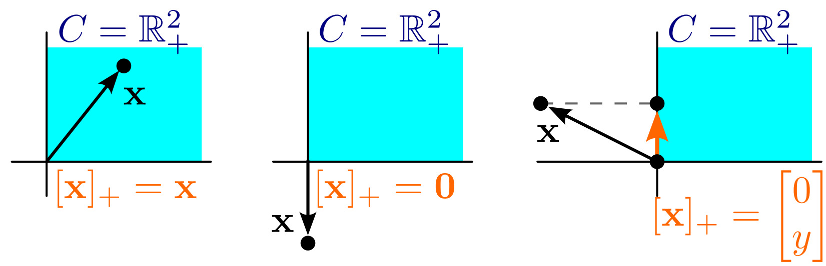
(i) Find an expression for the non-negative real part $[\mathbf{x}]_+$ for each drawn $\mathbf{x}$ in terms of $\mathbf{x} = (x_1, x_2)$.

(a) The point is already in the non-negative orthant, so $[\mathbf{x}]_+ = \mathbf{x}$.

(b) The point is in the negative orthant, so $[\mathbf{x}]_+ = \mathbf{0}$.

(c) x_1 *is negative and* x_2 *is positive, so* $[\mathbf{x}]_+ = (0, x_2)$.

(ii) Sketch $[\mathbf{x}]_+$ for each vector.


$C = \mathbb{R}^2_+$
$\mathbf{x}$
$[\mathbf{x}]_+ = \mathbf{x}$
$C = \mathbb{R}^2_+$
$\mathbf{x}$
$[\mathbf{x}]_+ = \mathbf{0}$
$C = \mathbb{R}^2_+$
$\mathbf{x}$
$[\mathbf{x}]_+ = \begin{bmatrix} 0 \\ y \end{bmatrix}$

Worksheet 8-2: Q3

1. Consider the set $C = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 \geq 0, y_2 \geq 0, y_3 = 0\}$. Write the orthogonal projection operator $P_C(\mathbf{x})$ for any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. What point in C minimizes $P_c(\mathbf{x})$? Write it in terms of $[-]_+$.

   - *The projection operator is* $P_C(\mathbf{x}) = \mathbf{y}$ *where* $\mathbf{y}$ *minimizes*

     $$\min_{\mathbf{y}} \|\mathbf{x} - \mathbf{y}\|^2$$

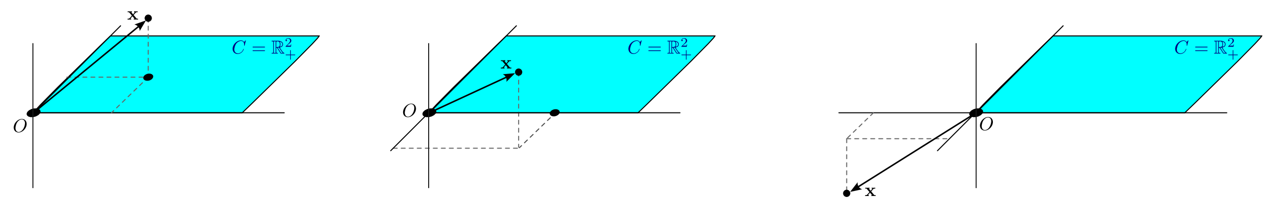
     which is equivalent to

     $$\min_{\mathbf{y}} (x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - 0)^2.$$

   - *This is separable, so the projection is the point* (y_1, y_2, y_3) *that minimizes* $(y_1 - x_1)^2$, $(y_2 - x_2)^2$, *and* $(x_3)^2$ *separately (since* $y_3 = 0$*).*
   - *This means the point in* C *that minimizes both functions is* $P_C(\mathbf{x}) = ([x_1]_+, [x_2]_+, 0)$.

2. For each of the following points, write the expression for the projected point in terms of just x_1, x_2, x_3. Sketch the point.

(a) (b) (c)



(a) Since $x_1, x_2, x_3 \geq 0$, *we have*

$$P_C(\mathbf{x}) = ([x_1]_+, [x_2]_+, 0) = (x_1, x_2, 0)$$

(b) Here, $x_1, x_3 \geq 0$, *but* $x_2 \leq 0$. *So we have*

$$P_C(\mathbf{x}) = ([x_1]_+, [x_2]_+, 0) = (x_1, 0, 0)$$

(c) In this case, $x_1, x_2, x_3 \leq 0$, *so*

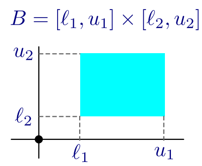
$$P_C(\mathbf{x}) = ([x_1]_+, [x_2]_+, 0) = (0, 0, 0)$$

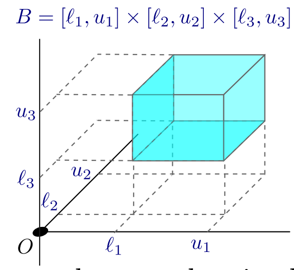
Worksheet 8-2: Q4

A box is a subset of $\mathbb{R}^n$ of the form

$$B = [\ell_1, u_1] \times [\ell_2, u_2] \times \ldots \times [\ell_n, u_n] \\ = \{\mathbf{x} \in \mathbb{R}^n : \ell_i \leq x_i \leq u_i\},$$

where $\ell_i \leq u_i$ for all $i = 1, 2, \ldots, n$.





We will assume that some of the u_is can be ∞ and some of the ℓ_is can be $-\infty$; but in these cases we will assume that $-\infty$ or ∞ are not contained in the intervals. The figure above shows some two examples in boxes in $\mathbb{R}^2$ and $\mathbb{R}^3$. The orthognal projection on the box is the minimizer of the convex optimization problem

$$\begin{array}{ll} \min & \|\mathbf{y} - \mathbf{x}\|^2 \\ \text{s.t.} & \mathbf{y} \in B \end{array}$$

for a given box B.

(a) Write the minimization problem above in terms of only x_i's and y_i's.

$$\begin{array}{ll} \mathit{min} & \sum_{i=1}^{n} (y_i - x_i)^2 \\ \mathit{s.t.} & y_i \in [\ell_i, u_i] \text{ for each } i \end{array}$$

(b) Is the resulting functional equation separable? Justify your answer.

Yes it's separable. This is exactly the definition since I have the function in pieces that each only depend on one of the constraints.

(c) Write down and solve the optimization problem for each y_i. Use this to determine $\mathbf{y} = P_C(\mathbf{x})$.

The problem becomes

$$\min\{(y_i - x_i)^2 \mid y_i \in [\ell_i, u_i]\}$$

which has value 0 if $x_i \in [\ell_i, u_i]$ since we just set $y_i = x_i$. On the other hand, if $x_i < \ell_i$, the minimum occurs at the lower bound $y_i = \ell_i$, and similarly if $x_i > u_i$, it occurs at $y_i = u_i$. Putting this together, the optimum is at

$$y_i = \begin{cases} u_i, & x_i \geq u_i, \\ x_i, & \ell_i < x_i < u_i, \\ \ell_i & x_i \leq \ell_i, \end{cases}$$

and so the full solution is $\mathbf{y} = (y_1, \cdots, y_n)$ for the y_i's given above.

Worksheet 8-2: Q5

Consider the normal ball in $\mathbb{R}^2$, $C = B[0, r] = \{\mathbf{y} = (y_1, y_2) \mid \|\mathbf{y}\|_2 \le r\}$. We will find the projection $P_C(\mathbf{x})$ for some point $\mathbf{x} \in \mathbb{R}^2$, which is the $\mathbf{y}$ that minimizes

$$\begin{aligned} \min \quad & \|\mathbf{y} - \mathbf{x}\|^2 \\ \text{s.t.} \quad & \|\mathbf{y}\|^2 \le r^2. \end{aligned}$$

(a) Assume $\mathbf{x} \in B[0, r]$. What is $P_C(\mathbf{x})$ and why?

$P_C(\mathbf{x}) = \mathbf{x}$ since $\|x \le r^2$ as it's in $B[0, r]$, so then the function becomes

$$\|\mathbf{y} - \mathbf{x}\|^2 = \|\mathbf{x} - \mathbf{x}\|^2 = 0$$

and this function is always ≥ 0, so this must be the minimum.

(b) What is $\nabla f(\mathbf{z})$ for $f(\mathbf{z}) = \|\mathbf{z}\|^2$?

This works for any dimension, but we're just focused on $\mathbb{R}^2$ right now. So think of $f(z_1, z_2) = z_1^2 + z_2^2$ to write the gradient as

$$\nabla f(\mathbf{z}) = \begin{bmatrix} 2z_1 \\ 2z_2 \end{bmatrix} = 2 \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 2\mathbf{z}$$

(c) Now we can deal with the case where $\mathbf{x} \notin B[0, r]$, equivalently $\|\mathbf{x}\|^2 \ge r^2$. We know (from the first order optimality condition for local optima, Thm 2.6 in the book) that if $\mathbf{x}^* \in \mathrm{int}(C)$ is a local optimum and all partial derivatives exist, then $\nabla f(\mathbf{x}^*) = 0$. If $\mathbf{x} \notin B[0, r]$ and somehow $\mathbf{x}^* = P_C(\mathbf{x}) \in \mathrm{int}(B[0, r])$, use your calculated gradient above to conclude that the result is impossible so $P_C(\mathbf{x})$ must be on the boundary of $B[0, r]$.

If $\mathbf{x}^* = P_C(\mathbf{x}) \in$ int$(B[0, r])$, the theorem mentioned says that we must have $\nabla f(\mathbf{x}^*) = 0$. But by the calculation above, this is the optimum of $f(\mathbf{x} - \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|^2$, so $\nabla f = 2(\mathbf{x}^* - \mathbf{x}) =$, but then $\mathbf{x}^* = \mathbf{x}$. The problem is $\mathbf{x} \notin B[0, r]$ but $\mathbf{x}^* \in B[0, r]$, so they can't be the same point. This means that our assumption that $\mathbf{x}^*$ was in the interior of $B[0, r]$ must be wrong, so $\mathbf{x}^*$ is on the boundary.

(d) By the previous, we know that if $\|\mathbf{x}\| \geq r$, the solution must be on the boundary, so we can replace the problem with

$$\begin{aligned} \min \quad & \|\mathbf{y} - \mathbf{x}\|^2 \\ \text{s.t.} \quad & \|\mathbf{y}\|^2 = r^2. \end{aligned}$$

Then I can expand $\|\mathbf{y} - \mathbf{x}\|^2 = \|\mathbf{y}\|^2 - 2\mathbf{y}^\top \mathbf{x} + \|\mathbf{x}\|^2$ and replace this problem with

$$\begin{aligned} \min \quad & \|\mathbf{y}\|^2 - 2\mathbf{y}^\top \mathbf{x} + \|\mathbf{x}\|^2 \\ \text{s.t.} \quad & \|\mathbf{y}\|^2 = r^2. \end{aligned}$$

Why, then, can I replace this problem with the following problem?

$$\begin{aligned} \min \quad & -2\mathbf{y}^\top \mathbf{x} \\ \text{s.t.} \quad & \|\mathbf{y}\|^2 = r^2. \end{aligned}$$

We know that $\|y\|^2 = r^2$ is a constant that doesn't depend on $\mathbf{y}$. Also, $\|\mathbf{x}\|^2$ doesn't depend on $\mathbf{y}$. So $\|\mathbf{y}\|^2 + \|\mathbf{x}\|^2$ is a constant as far as $\mathbf{y}$ is concerned. So the optimization will find a minimum at the same $\mathbf{y}$ whether we use

$$\|\mathbf{y} - \mathbf{x}\|^2 = -2\mathbf{y}^\top \mathbf{x} + \mathit{constant}$$

or just $-2\mathbf{y}^\top \mathbf{x}$.

(e) The Cauchy-Schwartz inequality ($|\mathbf{u}^\top \mathbf{v}| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$) gives us a bound

$$\mathbf{y}^\top \mathbf{x} \leq |\mathbf{y}^\top \mathbf{x}| \leq \|\mathbf{y}\| \cdot \|\mathbf{x}\|$$

Use the above to justify each inequality below.

$$-2\mathbf{y}^\top \mathbf{x} \geq -2\|\mathbf{y}\|\|\mathbf{x}\| = -2r\|\mathbf{x}\|. \tag{1}$$

The first inequality comes from multiplying the top equation by -2 and reversing the inequality because it's negative. The second equality is because $\mathbf{y}$ is on the boundary of the ball, so $\|\mathbf{y}\| = r$.

(f) Check that equality in Eqn. 1 (meaning $-2\mathbf{y}^\top\mathbf{x} = -2r\|\mathbf{x}\|$) occurs when $\mathbf{y}^* = r\frac{\mathbf{x}}{\|\mathbf{x}\|}$.

Notice we can just drop the -2 *on each side to see if* $\mathbf{y}^\top\mathbf{x} = r\|\mathbf{x}\|$. *Plugging in* $\mathbf{y} = \mathbf{y}^*$ *on the left gives*

$$\mathbf{y}^\top\mathbf{x} = \frac{r}{\|\mathbf{x}\|}\mathbf{x}^\top\mathbf{x} = \frac{r}{\|\mathbf{x}\|}\|\mathbf{x}\|^2 = r\|\mathbf{x}\|.$$

(g) Check that $\mathbf{y}^* = r\frac{\mathbf{x}}{\|\mathbf{x}\|}$ is in $B[0, r]$.

In this case, we just need to be sure it is on the boundary, so I need $\|\mathbf{y}^*\|^2 = r^2$. *But we can check this since*

$$\|\mathbf{y}^*\|^2 = \left\|r\frac{\mathbf{x}}{\|\mathbf{x}\|}\right\|^2 = \frac{r^2}{\|\mathbf{x}\|^2}\|\mathbf{x}\|^2 = r^2$$

like we wanted.

(h) Putting the above together, fill in the orthogonal projection for the ball:

$$P_{B[0,r]} = \begin{cases} \boxed{\mathbf{x}} & \|\mathbf{x}\| \leq r \\ \boxed{r\frac{\mathbf{x}}{\|\mathbf{x}\|}} & \|\mathbf{x}\| > r \end{cases}$$