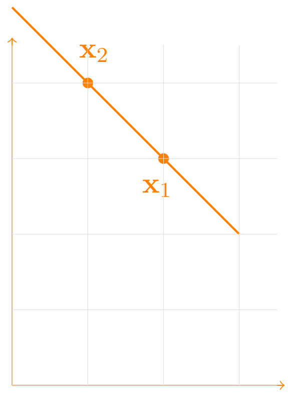
Name: **Present group members:**

Worksheet 6-1: Q1

Consider the points $\mathbf{x}_1 = (2,3)$ and $\mathbf{x}_2 = (1,4)$ in $\mathbb{R}^2$.

1. Sketch the points $\mathbf{x}_1$ and $\mathbf{x}_2$ in the plane and draw the line through them. For parts 3-6 below, sketch the point on the line corresponding to each value of λ.



2. Write the equation for the line through the two points using the equation $L(\lambda) = \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$ for $\lambda \in \mathbb{R}$. *The line through the two points is given by* $\lambda(2,3) + (1-\lambda)(1,4)$ *for* $\lambda \in \mathbb{R}$.

3. What point is on the line when $\lambda = 0$? $x_2 = (1,4)$

4. What point is on the line when $\lambda = 1$? $x_1 = (2,3)$

5. What point is on the line when $\lambda = 0.5$? $\frac{1}{2}x_1 + \frac{1}{2}x_2 = (1.5, 3.5)$

6. What point is on the line when $\lambda = -1$? $-x_1 + 2x_2 = (0,5)$. *Note that this one is outside the line segment connecting* x_1 *and* x_2*, but it's still on the line.*

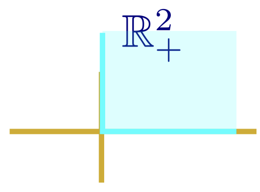
Worksheet 6-1: Q2

For each of the following, indicate whether the given set is convex or not and justify your answer.

i) $S = \{(1,2),(3,4),(4,5),(6,7)\} \subset \mathbb{R}^2$

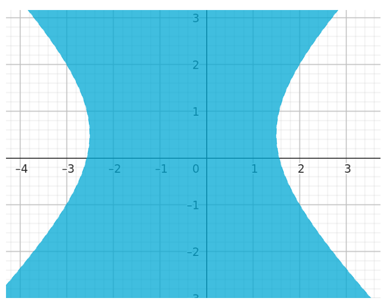
Since the line connecting any two elements is not in the set, S is not convex.

ii) $S \subseteq \mathbb{R}^2_+ = \{(x_1,x_2) \mid x_1,x_2 \geq 0\}$ where for any $(x,y) \in S$ we have $xy = 0$.



If $xy = 0$, then either $x = 0$ or $y = 0$. So the set is the "L"-shaped set composed of the x and the y axis. Any line connecting any point on the x-axis contains points not on the x and y axis. So this set is not convex.

iii) $S = \{\mathbf{x} \in \mathbb{R}^2 \mid x_1^2 - x_2^2 + x_1 + x_2 \leq 4\}$ shown below.



The set is not convex. Take any vertical line on the left or right side, say from $(2,-1)$ to $(2,2)$ and notice that the stuff in the middle is not in the set.

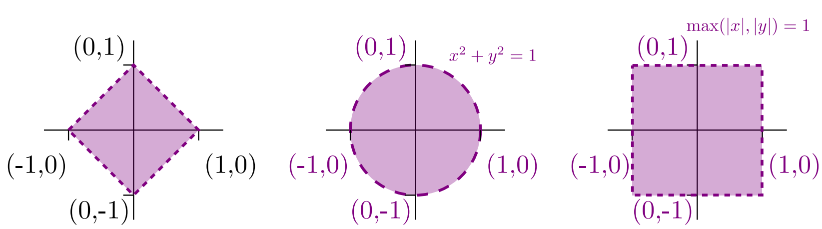
iv) $\{\mathbf{x} : \|x\|_1 \leq 1\}$. L_1-norm: $\|\mathbf{x}\|_1 := \sum_{i=1}^n |x_i|$ in Fig. (a) below.

This is convex.

v) $\{\mathbf{x} : \|x\|_2 \leq 1\}$. L_2-norm: $\|\mathbf{x}\|_2 := \sqrt{\sum_{i=1}^n x_i^2}$ shown in Fig. (b).

This is convex.

vi) $\{\mathbf{x} : \|x\|_\infty \leq 1\}$. L_∞-norm: $\|\mathbf{x}\|_\infty := \max_i |x_i|$ shown in in Fig. (c). *This is convex.*



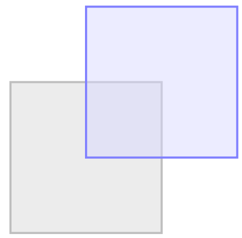
(a) L_1: $\|x\|_1 = 1$ (b) L_2: $\|x\|_2 = 1$ (c) L_∞: $\|x\|_\infty = 1$

Worksheet 6-1: Q3

The intersection of convex sets is convex, but the **union** of convex sets is not necessarily convex.

1. Draw a simple example to of 2 or 3 convex sets where the union is not convex.

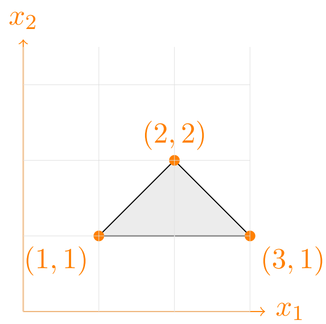
   Some examples: This could be two ovals in an L-shape, or two circles that are far apart.



2. Draw a simple example of 2 or 3 convex sets where the union is convex. *Some examples: This could be two circles that are close together, or a circle and a square that overlap.*

Worksheet 6-1: Q4

1. Sketch the points in the set $S = \{(1,1),(2,2),(3,1)\}$ and the convex hull of S in the plane.



2. A convex combination of k points is written as $\lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2 + \ldots + \lambda_k\mathbf{x}_k$ where $\lambda_i \geq 0$ for all i and $\sum_{i=1}^{k} \lambda_i = 1$. What are the λ_1, λ_2, and λ_3 values for each of the following points in the convex hull of S?

   i) $(2,2)$ *(0,1,0)*
   ii) $(3,1)$ *(0,0,1)*
   iii) $(1.5,1.5)$ *(0.5,0.5,0)*
   iv) $(2,1)$ *(0.5,0,0.5)*

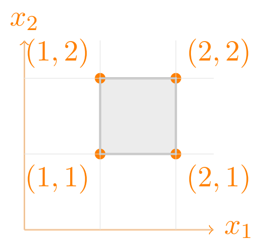
3. What points are represented by the following choices of λ_k for convex combinations of points in S?

   (a) $\lambda_1 = \frac{1}{3}$, $\lambda_2 = \frac{1}{3}$, $\lambda_3 = \frac{1}{3}$ $(2, 4/3)$
   (b) $\lambda_1 = \frac{1}{2}$, $\lambda_2 = \frac{1}{4}$, $\lambda_3 = \frac{1}{4}$ $(2.25, 1.5)$
   (c) $\lambda_1 = \frac{1}{2}$, $\lambda_2 = 0$, $\lambda_3 = \frac{1}{2}$ $(1.5, 1.5)$

Worksheet 6-1: Q5
Let S contains four points

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

1. Sketch the points in S and the convex hull of S in the plane.



2. How many points in S do we need at most to express any point $\mathbf{x}$ in the convex hull as the convex sum of those points?

   Recall, Theorem: If $x \in Conv(S) \subset \mathbb{R}^d$, then x is the convex sum of at most $d+1$ points of S.

   In this example, $d = 2$, so we can express $\mathbf{x}$ as the sum of at most 3 points in S.

3. Given $\mathbf{x} = \begin{bmatrix} 1.5 \\ 1.4 \end{bmatrix}$, write $\mathbf{x}$ as a convex combination of points in S using the number of points you said you needed above.

   One way to write $\mathbf{x}$ as a convex combination of points in S is:

$$\mathbf{x} = 0.1\mathbf{x}_1 + 0.4\mathbf{x}_2 + 0.5\mathbf{x}_3.$$

   Note that $0.1 + 0.4 + 0.5 = 1$, so this is a valid convex combination.