# Convex Sets: Part 1

## Lecture 6-1 - CMSE 382

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# This Lecture

**Topics:**

- Definition of convex sets
- Algebraic operations on Convex Sets
- Convex Hull

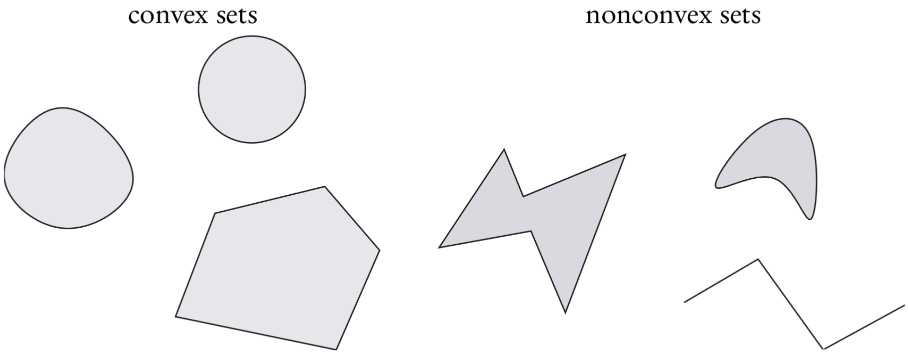
**Announcements:**

- None

Section 1

# Convex Set Definitions

A set $C \subseteq \mathbb{R}^n$ is **convex** if for any $\mathbf{x}, \mathbf{y} \in C$, the line segment $[\mathbf{x}, \mathbf{y}]$ is also in $C$.



# Examples: Lines

A line in $\mathbb{R}^n$ is a set of points in the form

$$L = \{\mathbf{z} + t\mathbf{d} : t \in \mathbb{R}\},$$

where $\mathbf{z}, \mathbf{d} \in \mathbb{R}^n$ and $\mathbf{d} \neq 0$. The point $\mathbf{z}$ is on the line, and the line is parallel to the direction $\mathbf{d}$.

Closed and open line segments between two different points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$:

- $[\mathbf{x}, \mathbf{y}] := \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} : 0 \leq \lambda \leq 1\}$
- $(\mathbf{x}, \mathbf{y}) := \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} : 0 < \lambda < 1\}$
- $[\mathbf{x}, \mathbf{y}) := \{\lambda\mathbf{x} + (1-\lambda)\mathbf{y} : 0 < \lambda \leq 1\}$

desmos.com/calculator/2ijmw3zike

# More Examples

- empty set $\emptyset$ and the whole space $\mathbb{R}^n$.
- hyperplane: $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\top \mathbf{x} = b\}$ for $\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
- closed and open half-spaces: Given $\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$
  - $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\top \mathbf{x} \leq b\}$
  - $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^\top \mathbf{x} < b\}$
- balls: Given $\mathbf{c} \in \mathbb{R}^n$ and $r > 0$
  - $B(\mathbf{c}, r) := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{c}\| < r\}$
  - $B[\mathbf{c}, r] := \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x} - \mathbf{c}\| \leq r\}$.
- ellipsoid: $\{\mathbf{x} \in \mathbb{R}^n : (\mathbf{x} - \mathbf{b})^\top \mathbf{Q}(\mathbf{x} - \mathbf{b}) \leq c\}$ with $\mathbf{Q} \in \mathbb{R}^{n \times n}$ being positive semidefinite, $\mathbf{b} \in \mathbb{R}^b$, and $c > 0$.

Section 2

# Algebraic Operations on Convex Sets

# Intersection of Convex Sets

## Property

The intersection of any convex sets is convex.

## Example

Convex polytopes $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}_i x \leq \mathbf{b}_i\}$ with $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$.

# Linear Combinations

**Property**

The linear combination of a finite number of convex sets is convex

- Let $C_1, C_2, \ldots, C_k \subseteq \mathbb{R}^n$ be convex sets
- Let $\mu_1, \mu_2, \ldots, \mu_k \in \mathbb{R}$. Then the set

$$\mu_1 C_1 + \mu_2 C_2 + \ldots + \mu_k C_k$$
$$= \left\{ \sum_{i=1}^{k} \mu_i \mathbf{x}_i \mid \mathbf{x}_i \in C_i, i = 1, 2, \ldots, k \right\}$$

is convex.

# Products

**Property**

The Cartesian product of a finite number of convex sets is convex

- Let $C_i \subseteq \mathbb{R}^{k_i}$ be a convex set for any $i = 1, 2, \ldots, m$.
- Then

$$\begin{aligned} C_1 \times C_2 \times \ldots \times C_m \\ = \{(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m) \mid \mathbf{x}_i \in C_i, i = 1, 2, \ldots, m\} \end{aligned}$$

is convex.

**Property**

The image of a convex set under a linear transformation is convex.

Let $M \subseteq \mathbb{R}^n$ be a convex set and let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then the set

$$\mathbf{A}(M) = \{\mathbf{A}\mathbf{x} \mid \mathbf{x} \in M\}$$

is convex.

**Property**

If a convex set is a linear transform of another set, then that set is convex

Let $D \subseteq \mathbb{R}^m$ be a convex set, and let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then the set

$$\mathbf{A}^{-1}(D) = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \in D\}$$

is convex.

# Nonexample

Unions of convex sets are not necessarily convex.

Section 3

# Topological Properties of Convex Sets

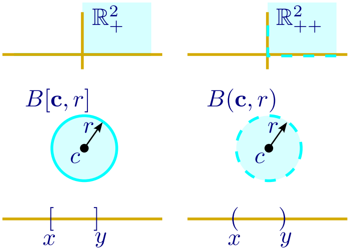
# Recall: Interior points

## Definition

Given a set $U \subset \mathbb{T}$, a point $\mathbf{c} \in U$ is an interior point of $U$ if there exists $r > 0$ such that $B(\mathbf{c}, r) \subset U$. The set of all interior points is called the **interior** and defined as $\text{int}(U) = \{\mathbf{x} \in U : B(\mathbf{c}, r) \subset U \text{ for some } r > 0\}$.

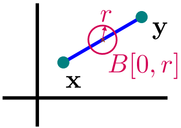
Examples:

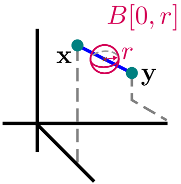
- $\text{int}(\mathbb{R}^n_+) = \mathbb{R}^n_{++}$
- $\text{int}(B[\mathbf{c}, r]) = B(\mathbf{c}, r)$
- $\text{int}([\mathbf{x}, \mathbf{y}]) = (\mathbf{x}, \mathbf{y}) \quad \text{if } \mathbf{x} \neq \mathbf{y}$
- $\text{int}([\mathbf{x}, \mathbf{x}]) = \emptyset$



# Examples of convex sets with empty interior

**Line segment in high dimensional ambient space**

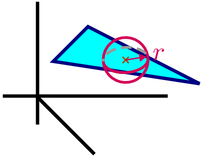


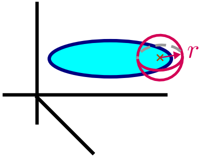

$[\mathbf{x}, \mathbf{y}] := \{\mathbf{x} + \alpha(\mathbf{y} - \mathbf{x}) : \alpha \in [0, 1]\}$

$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $n > 1$

**Flat polygon in $\mathbb{R}^n$, $n > 2$ ambient space**




**Flat disk in $\mathbb{R}^n$, $n > 2$ ambient space**



# Topological Properties of Convex Sets

**Theorem**

*The closure of a convex set is convex.*

**Theorem**

*The interior of a convex set is convex.*

**Theorem**

*Let $C$ be a convex set with a nonempty interior. Then:*

1. $cl(int(C)) = cl(C)$
2. $int(cl(C)) = int(C)$

**Theorem**

*The convex hull of a finite set is closed.*

Section 4

# Convex Hulls

# Convex Combinations

## Definition

Given $k$ points $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k \in \mathbb{R}^n$,
a **convex combination** of these $k$ points is a point of the form

$$\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2 + \cdots + \lambda_k \mathbf{x}_k$$

where $\lambda_1, \lambda_2, \ldots, \lambda_k$ are nonnegative numbers satisfying $\lambda_1 + \lambda_2 + \cdots + \lambda_k = 1$. We denote

$$\Delta_m = \{(\lambda_1, \lambda_2, \cdots, \lambda_m) \mid \lambda_i \geq 0,\ \forall i,\ \sum_{i=1}^{m} \lambda_i = 1\}.$$

A convex set is defined by the property that any convex combination of two points from the set is also in the set.

# Convex Combinations

**Theorem:** Let $C \subset \mathbb{R}^n$ be a convex set and $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m \in C$. Then for any $\lambda \in \Delta_m$, we have $\sum_{i=1}^m \lambda_i \mathbf{x}_i \in C$.

- A convex combination of any number of points from a convex set is in the set.

# Convex Hulls

Let $S \subset \mathbb{R}^n$. The **convex hull** of $S$ is defined as

$$\operatorname{conv}(S) := \left\{ \sum_{i=1}^{k} \lambda_i \mathbf{x}_i \mid \mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_k \in S, \lambda \in \Delta_k, k \in \mathbb{N} \right\}$$

**Lemma:** Let $S \subseteq \mathbb{R}^n$. If $S \subseteq T$ for some convex set $T$, then $\operatorname{conv}(S) \subseteq T$.

- The convex hull of $S$ is the smallest convex set containing $S$.

**Theorem:** If $x \in \mathrm{Conv}(S) \subset \mathbb{R}^d$, then $x$ is the convex sum of at most $d+1$ points of $S$.

# Example

Let $S$ contains four points

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \mathbf{x}_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \mathbf{x}_4 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

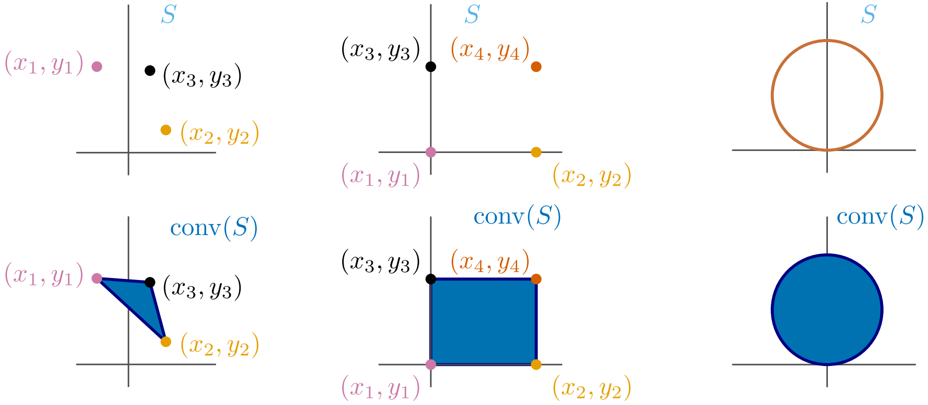
$\text{conv}(S) = \{(x_1, x_2) : 1 \leq x_1 \leq 2; 1 \leq x_2 \leq 2\}.$

Given $\mathbf{x} = \begin{bmatrix} 1.5 \\ 1.4 \end{bmatrix}$, we can find three points from the four points such that $\mathbf{x}$ is a linear combination of these three points.

- $\mathbf{x}$ is a linear combination of $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$:

$$\mathbf{x} = 0.1\mathbf{x}_1 + 0.4\mathbf{x}_2 + 0.5\mathbf{x}_3$$

# Convex Hull Examples



# Groups - Round 3

**Group 1**
Lowell, Tianjuan, Lauryn, Atticus

**Group 2**
Alice, Aidan, Dev, Anthony

**Group 3**
Abigail, Michal, Breena, Andrew

**Group 4**
Kyle, Vinod, Dori, Joseph

**Group 5**
Yousif, Jamie, Jay, K.M Tausif

**Group 6**
Shanze, Saitej, Karen, Jack

**Group 7**
Arjun, Noah, Luis, Arya

**Group 8**
Morgan, Jonid, Sanskaar, Jake

**Group 9**
Quang Minh, Monirul Amin, Daniel, Ha

**Group 10**
Braedon, Dominic, Zheng, Lora

**Group 11**
Sai, Brandon, Purvi, Aaron

**Group 12**
Igor, Scott, Maye, Long