Name: **Present group members:**

Worksheet 11-1: Q1

Consider the problem

$$\begin{aligned} \min \quad & x_1^2 - 2x_2 \\ \text{s.t.} \quad & x_1^2 + x_2^2 = 4 \end{aligned}$$

(a) On this Desmos plot, what are the purple surface, and the orange and green curves? Visually on the plot, what should the answer to the optimization problem be?

(b) If you couldn't see the plot, how would you justify that the problem has an optimal solution?

(c) What are the regular points of the problem? Use this to conclude that the optimal solution will be one of the KKT points.

(d) Write down the KKT constraint equations.

(e) Find the KKT points. *Hint: There are four of them.*

(f) What are the function values at the KKT points? Which one is the optimal solution? Plot the two KKT points on the Desmos plot to verify your answer.

Worksheet 11-1: Q2
Consider the problem

$$\begin{aligned} \min \quad & 2x_1 + 3x_2 - x_3 \\ \text{s.t.} \quad & x_1^2 + x_2^2 + x_3^2 = 1 \\ & x_1^2 + 2x_2^2 + 2x_3^2 = 2. \end{aligned}$$

(a) Do you expect the problem to have an optimal solution? Explain.

(b) Write down all the KKT constraint equations.

(c) Use your stationarity conditions to check that $\mu_1 + \mu_2 \neq 0$ and $\mu_1 + 2\mu_2 \neq 0$.

(d) Write $a = \frac{1}{\mu_1+\mu_2}$ and $b = \frac{1}{\mu_1+2\mu_2}$. Use the stationarity conditions to express any KKT point $\mathbf{x} = (x_1, x_2, x_3)$ in terms of a and b.

(e) Plug this point into the two constraint equations. Can any KKT point satisfy the constraints?

From part (a), you hopefully came to the conclusion that an optimal solution must exist. From part (e), we know there are no KKT points. This means the optimal solution must be an irregular point.

Here we have two constraints h_1 and h_2, so we have two gradient vectors. Two vectors are linearly dependent if one is a scalar multiple of the other, that is, if

$$\nabla h_1(x_1, x_2, x_3) = \alpha \nabla h_2(x_1, x_2, x_3).$$

for some $\alpha \neq 0$.

There are two cases in this problem where this can happen.

(f) If $x_1 = 0$, check that the two gradients are linearly dependent. What possible values of x_2 and x_3 can we have in this case?

(g) Rewrite the original problem replacing $x_1 = 0$ which should now give you a problem in two variables with one constraint.

(h) Find the solution for the optimization problem you set up in the last question where we assumed $x_1 = 0$, and use this to give an irregular point to be checked for the solution of the original problem.

(i) The other case where there can be linear dependence is if $x_2 = x_3 = 0$. Check that the two gradients of the two constraint functions are linearly dependent in this case and give all possible irregular points of this form.

(j) Plug these possible points into the original constraints. Are there any irregular points of this form that satisfy the constraints?

(k) Putting all of this together, what is the solution to the original problem?