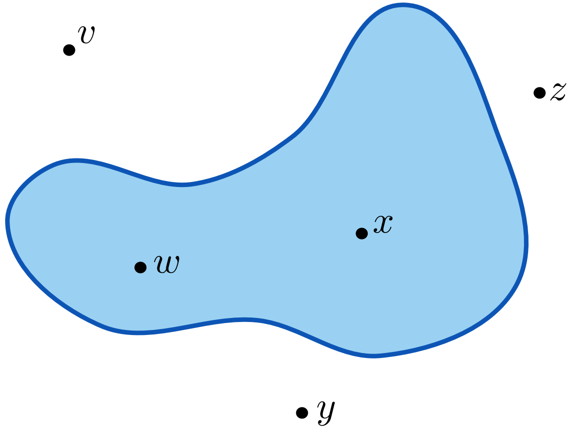
Name: **Present group members:**

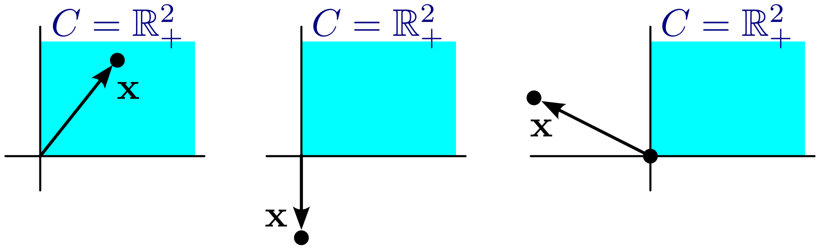
Worksheet 8-2: Q1

For the following sets and each point drawn (v, w, x, y, and z), mark the point that minimizes the projection operator ($P_C(v)$, $P_C(w)$, $P_C(x)$, $P_C(y)$, and $P_C(z)$).



Worksheet 8-2: Q2

For each of the shown vectors $\mathbf{x}$, answer the following



(a) (b) (c)

(i) Find an expression for the non-negative real part $[\mathbf{x}]_+$ for each drawn $\mathbf{x}$ in terms of $\mathbf{x} = (x_1, x_2)$.

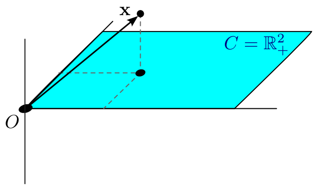
(ii) Sketch $[\mathbf{x}]_+$ for each vector.

Worksheet 8-2: Q3

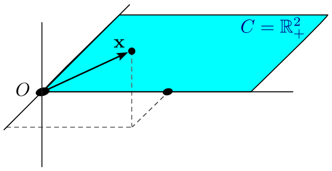
1. Consider the set $C = \{(y_1, y_2, y_3) \in \mathbb{R}^3 \mid y_1 \geq 0, y_2 \geq 0, y_3 = 0\}$. Write the orthogonal projection operator $P_C(\mathbf{x})$ for any $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. What point in C minimizes $P_c(\mathbf{x})$? Write it in terms of $[-]_+$.

2. For each of the following points, write the expression for the projected point in terms of just x_1, x_2, x_3. Sketch the point.

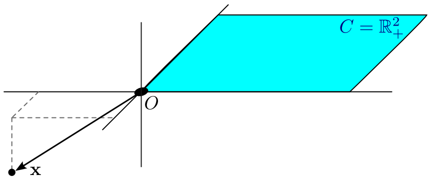
(a)



(b)



(c)

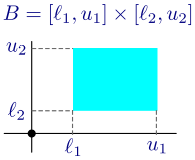


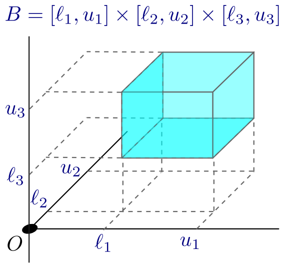
Worksheet 8-2: Q4

A box is a subset of $\mathbb{R}^n$ of the form

$$\begin{aligned} B &= [\ell_1, u_1] \times [\ell_2, u_2] \times \ldots \times [\ell_n, u_n] \\ &= \{\mathbf{x} \in \mathbb{R}^n : \ell_i \leq x_i \leq u_i\}, \end{aligned}$$

where $\ell_i \leq u_i$ for all $i = 1, 2, \ldots, n$.





We will assume that some of the u_is can be ∞ and some of the ℓ_is can be $-\infty$; but in these cases we will assume that $-\infty$ or ∞ are not contained in the intervals. The figure above shows some two examples in boxes in $\mathbb{R}^2$ and $\mathbb{R}^3$. The orthognal projection on the box is the minimizer of the convex optimization problem

$$\begin{array}{ll} \min & \|\mathbf{y} - \mathbf{x}\|^2 \\ \text{s.t.} & \mathbf{y} \in B \end{array}$$

for a given box B.

(a) Write the minimization problem above in terms of only x_i's and y_i's.

(b) Is the resulting functional equation separable? Justify your answer.

(c) Write down and solve the optimization problem for each y_i. Use this to determine $\mathbf{y} = P_C(\mathbf{x})$.

Worksheet 8-2: Q5

Consider the normal ball in $\mathbb{R}^2$, $C = B[0, r] = \{\mathbf{y} = (y_1, y_2) \mid \|\mathbf{y}\|_2 \leq r\}$. We will find the projection $P_C(\mathbf{x})$ for some point $\mathbf{x} \in \mathbb{R}^2$, which is the $\mathbf{y}$ that minimizes

$$\begin{aligned} \min \quad & \|\mathbf{y} - \mathbf{x}\|^2 \\ \text{s.t.} \quad & \|\mathbf{y}\|^2 \leq r^2. \end{aligned}$$

(a) Assume $\mathbf{x} \in B[0, r]$. What is $P_C(\mathbf{x})$ and why?

(b) What is $\nabla f(\mathbf{z})$ for $f(\mathbf{z}) = \|\mathbf{z}\|^2$?

(c) Now we can deal with the case where $\mathbf{x} \notin B[0, r]$, equivalently $\|\mathbf{x}\|^2 \geq r^2$. We know (from the first order optimality condition for local optima, Thm 2.6 in the book) that if $\mathbf{x}^* \in \text{int}(C)$ is a local optimum and all partial derivatives exist, then $\nabla f(\mathbf{x}^*) = 0$. If $\mathbf{x} \notin B[0, r]$ and somehow $\mathbf{x}^* = P_C(\mathbf{x}) \in \text{int}(B[0, r])$, use your calculated gradient above to conclude that the result is impossible so $P_C(\mathbf{x})$ must be on the boundary of $B[0, r]$.

(d) By the previous, we know that if $\|\mathbf{x}\| \geq r$, the solution must be on the boundary, so we can replace the problem with

$$\begin{array}{ll} \min & \|\mathbf{y}-\mathbf{x}\|^2 \\ \text{s.t.} & \|\mathbf{y}\|^2 = r^2. \end{array}$$

Then I can expand $\|\mathbf{y}-\mathbf{x}\|^2 = \|\mathbf{y}\|^2 - 2\mathbf{y}^\top\mathbf{x} + \|\mathbf{x}\|^2$ and replace this problem with

$$\begin{array}{ll} \min & \|\mathbf{y}\|^2 - 2\mathbf{y}^\top\mathbf{x} + \|\mathbf{x}\|^2 \\ \text{s.t.} & \|\mathbf{y}\|^2 = r^2. \end{array}$$

Why, then, can I replace this problem with the following problem?

$$\begin{array}{ll} \min & -2\mathbf{y}^\top\mathbf{x} \\ \text{s.t.} & \|\mathbf{y}\|^2 = r^2. \end{array}$$

(e) The Cauchy-Schwartz inequality ($|\mathbf{u}^\top\mathbf{v}| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$) gives us a bound

$$\mathbf{y}^\top\mathbf{x} \leq |\mathbf{y}^\top\mathbf{x}| \leq \|\mathbf{y}\| \cdot \|\mathbf{x}\|$$

Use the above to justify each inequality below.

$$-2\mathbf{y}^\top\mathbf{x} \geq -2\|\mathbf{y}\|\|\mathbf{x}\| = -2r\|\mathbf{x}\|. \tag{1}$$

(f) Check that equality in Eqn. 1 (meaning $-2\mathbf{y}^\top\mathbf{x} = -2r\|\mathbf{x}\|$) occurs when $\mathbf{y}^* = r\frac{\mathbf{x}}{\|\mathbf{x}\|}$.

(g) Check that $\mathbf{y}^* = r\frac{\mathbf{x}}{\|\mathbf{x}\|}$ is in $B[0, r]$.

(h) Putting the above together, fill in the orthogonal projection for the ball:

$$P_{B[0,r]} = \begin{cases} \boxed{} & \|\mathbf{x}\| \leq r \\ \\ \boxed{} & \|\mathbf{x}\| > r \end{cases}$$