Duality

Lecture 12-3 - CMSE 382

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Weds, April 15, 2026

This Lecture

Topics:

- Dual for strictly convex quadratic programming - Dual for convex quadratic programming

Announcements:

- Homework 6 is due on Friday, April 17, 2026 at 11:59pm.

Section 1

Last Time

Dual objective function

Consider the general model referred to as the primal model

$$
\begin{aligned}
f^* = \min \quad & f(\mathbf{x}) \\
\text{such that} \quad & g_i(\mathbf{x}) \le 0, i = 1, 2, \dots m, \\
& h_j(\mathbf{x}) = 0, j = 1, 2, \dots p, \\
& \mathbf{x} \in X, \ \text{where } X \subseteq \mathbb{R}^n,
\end{aligned}
$$

and f, g_i, h_j are functions defined on X.

The Lagrangian of the problem is

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \mu_j h_j(\mathbf{x}),$$

The dual objective function $q : \mathbb{R}_+^m \times \mathbb{R}^p \to \mathbb{R} \cup \{-\infty\}$ is

$$q(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \min_{x \in X} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}),$$

Weak duality theorem

Primal Problem

$$\begin{aligned} f^* = \min \quad & f(\mathbf{x}) \\ \text{such that} \quad & g_i(\mathbf{x}) \leq 0, i = 1, 2, \dots m, \\ & h_j(\mathbf{x}) = 0, j = 1, 2, \dots p, \\ & \mathbf{x} \in X, \text{ where } X \subseteq \mathbb{R}^n, \end{aligned}$$

and f, g_i, h_j are functions defined on X.

Dual Problem

$$\begin{aligned} q^* = \max \quad & q(\boldsymbol{\lambda}, \boldsymbol{\mu}) \\ \text{such that} \quad & (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathrm{dom}(q), \end{aligned}$$

where $\mathrm{dom}(q) = \{(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{R}^m_+ \times \mathbb{R}^p :$ $q(\boldsymbol{\lambda}, \boldsymbol{\mu}) > -\infty\}$, and $q(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \min_{\mathbf{x} \in X} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$.

Theorem (Weak duality theorem)

Consider the primal problem and its dual. Then $q^* \leq f^*$, where q^*, f^* are the optimal dual and primal values, respectively.

Desmos example

Strong duality of convex problems with equality & inequality constraints

Primal Problem (P)

$$
\begin{aligned}
f^* = \min \quad & f(\mathbf{x}) \\
\text{such that} \quad & g_i(\mathbf{x}) \leq 0, i = 1, 2, \ldots, m, \\
& h_j(\mathbf{x}) \leq 0, j = 1, 2, \ldots, p, \\
& s_k(\mathbf{x}) = 0, k = 1, 2, \ldots, q, \\
& \mathbf{x} \in X,
\end{aligned}
$$

Dual Problem (D)

$$
\begin{aligned}
q^* = \max \quad & q(\boldsymbol{\lambda}, \boldsymbol{\mu}) \\
\text{such that} \quad & (\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \text{dom}(q),
\end{aligned}
$$

where $\text{dom}(q) = \{(\boldsymbol{\lambda}, \boldsymbol{\mu}) \in \mathbb{R}_+^m \times \mathbb{R}^p : q(\boldsymbol{\lambda}, \boldsymbol{\mu}) > -\infty\}$, and $q(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \min_{\mathbf{x} \in X} L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu})$.

- For (P): X is a convex set and $f, g_1, \ldots, g_m$ are convex functions over X. The functions $h_1, \ldots, h_p, s_1, \ldots, s_q$ are affine.

Theorem (Strong duality under equality & inequality constraints)

If the generalized Slater's condition is satisfied in (P) and f^* has a finite optimal value, then the optimal value of (D) is attained, and the optimal values of the primal and dual problems are the same $f^* = q^*$.

Dual for linear programming

Primal

$$
\begin{aligned}
f^* = \min \quad & \mathbf{c}^\top \mathbf{x} \\
\text{such that} \quad & A\mathbf{x} \leq \mathbf{b},
\end{aligned}
$$

Dual

$$
\begin{aligned}
q^* = \max \quad & -\mathbf{b}^\top \boldsymbol{\lambda} \\
\text{such that} \quad & A^\top \boldsymbol{\lambda} = -\mathbf{c}, \\
& \boldsymbol{\lambda} \geq 0.
\end{aligned}
$$

Strong duality holds

If the primal problem is feasble (*meaning the constraint set is not empty*) and has a finite solution, then the optimal dual value is equal to the optimal primal value:

$$q^* = f^*.$$

Section 2

Quadratic Linear Programming

Strictly Convex Quadratic Programming

Primal

$$
\begin{aligned}
f^* = \min \quad & \mathbf{x}^\top Q\mathbf{x} + 2\mathbf{c}^\top \mathbf{x} \\
\text{such that} \quad & A\mathbf{x} \leq \mathbf{b},
\end{aligned}
$$

Dual

$$
\begin{aligned}
q^* = & \max q(\boldsymbol{\lambda}) \\
\text{such that} \quad & \boldsymbol{\lambda} \geq 0.
\end{aligned}
$$

- $Q \in \mathbb{R}^{n\times n}$ is **positive definite**, $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^m$.

- Strong duality holds
- $L(\mathbf{x}, \boldsymbol{\lambda}) = \mathbf{x}^T Q\mathbf{x} + 2\mathbf{c}^T\mathbf{x} + 2\boldsymbol{\lambda}^T(A\mathbf{x} - \mathbf{b})$
- $\nabla_x L(\mathbf{x}^*, \boldsymbol{\lambda}) = 2Q\mathbf{x}^* + 2(A^\top\boldsymbol{\lambda} + \mathbf{c}) = \mathbf{0}$
- $\mathbf{x}^* = -Q^{-1}(\mathbf{c} + A^\top\boldsymbol{\lambda})$

The objective function becomes

$$
\begin{aligned}
q(\boldsymbol{\lambda}) = -\boldsymbol{\lambda}^\top AQ^{-1}A^T\boldsymbol{\lambda} & \\
- 2(AQ^{-1}\mathbf{c} + \mathbf{b})^\top\boldsymbol{\lambda} & \\
- \mathbf{c}^\top Q^{-1}\mathbf{c}. &
\end{aligned}
$$

Dual for ~~strictly~~ convex quadratic programming

Convex Quadratic Program

$$\begin{aligned} \min \quad & \mathbf{x}^\top Q \mathbf{x} + 2\mathbf{c}^\top \mathbf{x} \\ \text{such that} \quad & A\mathbf{x} \leq \mathbf{b}, \end{aligned}$$

where $Q \in \mathbb{R}^{n \times n}$ is **positive semi-definite**, $\mathbf{c} \in \mathbb{R}^n$, and $\mathbf{b} \in \mathbb{R}^m$.

Since $Q \succeq 0$

- Q is not necessarily invertible
- The dual problem formulated for the strictly convex case is not possible in the convex case.
- A new formulation for the convex case is needed.
- The trick: write $Q = D^\top D$ for some matrix D, and make a new variable $\mathbf{z} = D\mathbf{x}$.

Dual for convex quadratic programming

Reformulated primal problem

$$\begin{aligned} \min \quad & \|\mathbf{z}\|^2 + 2\mathbf{c}^\top \mathbf{x} \\ \text{such that} \quad & A\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{z} = D\mathbf{x}. \end{aligned}$$

Dual Problem

$$\begin{aligned} \max \quad & -\|\boldsymbol{\mu}\|^2 - 2\mathbf{b}^\top \boldsymbol{\lambda} \\ \text{such that} \quad & \mathbf{c} + A^\top \boldsymbol{\lambda} - D^\top \boldsymbol{\mu} = 0, \\ & \boldsymbol{\lambda} \in \mathbb{R}^m_+, \boldsymbol{\mu} \in \mathbb{R}^n. \end{aligned}$$

Groups - Round 5

Group 1
Michal, Kyle, Daniel, Purvi

Group 2
Joseph, Jack, Scott, Breena

Group 3
Saitej, Dori, Noah, Tianjian

Group 4
Dev, Shanze, Lowell, Andrew

Group 5
Lora, Aidan, Arjun, Monirul Amin

Group 6
Anthony, Abigail, Atticus, Yousif

Group 7
Luis, Vinod, Morgan, Dominic

Group 8
Jay, Jonid, Alice, Aaron

Group 9
Arya, Jake, K M Tausif, Lauryn

Group 10
Maye, Ha, Zheng, Sai

Group 11
Jamie, Karen, Brandon, Quang Minh

Group 12
Long, Sanskaar, Braedon, Igor