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Present group members:

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**Worksheet 12-1: Q1**

Consider the problem

$$\begin{aligned} f^* &= \min x_1^2 - x_2 \\ \text{s.t. } &x_1 - x_2^2 = 0, \end{aligned}$$

1. Find the optimal solution by replacing  $x_1$  with  $x_2^2$  and making the objective a single variable function. What is the optimal value  $f^*$ ? (Note  $f^*$  is the minimal value of the objection function  $f(x_1, x_2) = x_1^2 - x_2$ , not the point  $x_1, x_2$  that achieves the minimum.)

*After replacement, the objective function becomes  $f(x_2) = x_2^4 - x_2$ . Take a derivative, we have  $f'(x_2) = 4x_2^3 - 1$ . Set  $f'(x_2) = 0$ , we have  $x_2 = \sqrt[3]{\frac{1}{4}}$ . So, the optimal value is*

$$f^* = -\frac{3}{4\sqrt[3]{4}} \approx -0.472.$$

2. Now solve the problem using duality:

*See [this desmos plot](#) for the visualization of the problem we discussed in class.*

- (a) Write down the Lagrangian.

*The Lagrangian is*

$$L(x_1, x_2, \mu) = x_1^2 - x_2 + \mu(x_1 - x_2^2).$$

- (b) What is  $\nabla_{x_1, x_2} L(x_1, x_2, \mu)$ ? When is  $\nabla_{x_1, x_2} L(x_1, x_2, \mu) = 0$ ?

*The gradient is*

$$\nabla_{x_1, x_2} L(x_1, x_2, \mu) = \begin{bmatrix} 2x_1 + \mu \\ -1 - 2\mu x_2 \end{bmatrix}.$$

*The gradient is zero when  $x_1 = -\frac{\mu}{2}$  and  $x_2 = -\frac{1}{2\mu}$ .*

- (c) Write a formula for  $q(\mu) = \min_{x_1, x_2} L(x_1, x_2, \mu)$  in terms of  $\mu$ .

*From above, we have that the Lagrangian is minimized at  $x_1 = -\frac{\mu}{2}$  and  $x_2 = -\frac{1}{2\mu}$ . This means the minimum value of the Lagrangian is*

$$\begin{aligned} L\left(-\frac{\mu}{2}, -\frac{1}{2\mu}, \mu\right) &= \left(-\frac{\mu}{2}\right)^2 - \left(-\frac{1}{2\mu}\right) + \mu\left(-\frac{\mu}{2} - \left(-\frac{1}{2\mu}\right)^2\right) \\ &= -\frac{\mu^2}{4} + \frac{1}{4\mu}. \end{aligned}$$

(d) What value of  $\mu$  maximizes  $q(\mu)$ ?

*Take a derivative, we have  $q'(\mu) = -\frac{\mu}{2} - \frac{1}{4\mu^2}$ . Set  $q'(\mu) = 0$ , we have*

$$\mu = -\sqrt[3]{\frac{1}{2}} = -2^{-1/3}.$$

(e) What is the optimal dual value  $q^* = \max_{\mu} q(\mu)$ ?

*The optimal dual value is*

$$q^* = q(-2^{-1/3}) = \frac{(-2^{-1/3})^2}{4} + \frac{1}{4(-2^{-1/3})} \approx -0.472$$

3. We know from the weak duality theorem that  $q^* \leq f^*$ . Is the bound tight for this problem? That is, do we have  $q^* = f^*$  in this problem?

*Yes, we have  $q^* = f^*$ , so the bound is tight.*

## Worksheet 12-1: Q2

Consider the problem

$$\begin{aligned} \min \quad & x_1^2 - x_2 \\ \text{s.t.} \quad & x_2^2 \leq 0, \end{aligned}$$

1. Find the optimal solution by reducing the problem to a single variable unconstrained optimization problem without using duality.

*The constraint  $x_2^2 \leq 0$  implies that  $x_2 = 0$ . So, the problem reduces to  $\min_{x_1} x_1^2$ , which has optimal solution  $x_1 = 0$  and optimal value  $f^* = 0$ .*

2. Write down the Lagrangian.

*The Lagrangian is*

$$L(x_1, x_2, \lambda) = x_1^2 - x_2 + \lambda x_2^2.$$

3. Obtain the dual objective function  $q(\lambda) = \min_{x_1, x_2} L(x_1, x_2, \lambda)$ . Note that you will need to consider the two cases:  $\lambda = 0$  and  $\lambda > 0$ . This means your objective function will be a piece-wise function.

*See [this desmos plot](#) for a visual of everything.*

*When  $\lambda = 0$ , the Lagrangian is just the objective function,*

$$L(x_1, x_2, 0) = x_1^2 - x_2.$$

*The gradient is*

$$\nabla_{x_1, x_2} L(x_1, x_2, 0) = \begin{bmatrix} 2x_1 \\ -1 \end{bmatrix}$$

*which is never zero. We also notice that the function goes to  $-\infty$  as  $x_2$  goes to  $\infty$ , so there is no finite minimum. So, we have  $q(0) = -\infty$ .*

*For  $\lambda > 0$ , the gradient of the Lagrangian is*

$$\nabla_{x_1, x_2} L(x_1, x_2, \lambda) = \begin{bmatrix} 2x_1 \\ -1 + 2\lambda x_2 \end{bmatrix}$$

*which is zero when  $x_1 = 0$  and  $x_2 = \frac{1}{2\lambda}$ . The value of the Lagrangian at this point is*

$$L\left(0, \frac{1}{2\lambda}, \lambda\right) = -\frac{1}{2\lambda} + \lambda \left(\frac{1}{2\lambda}\right)^2 = -\frac{1}{4\lambda}.$$

*So, we have*

$$q(\lambda) = \begin{cases} -\infty & \text{if } \lambda = 0 \\ -\frac{1}{4\lambda} & \text{if } \lambda > 0 \end{cases}.$$

4. What is  $q^* = \max_{\lambda \geq 0} q(\lambda)$ ?

*Since  $q(\lambda) = -\frac{1}{4\lambda}$  for  $\lambda > 0$ , we have  $q(\lambda) \rightarrow 0$  as  $\lambda \rightarrow \infty$ . So, we have  $q^* = \sup_{\lambda \geq 0} q(\lambda) = 0$ .*

5. Is the bound from the weak duality theorem tight for this problem? That is, do we have  $q^* = f^*$  in this problem?

*Yes, we have  $q^* = 0 = f^*$ , so the bound is tight.*

### Worksheet 12-1: Q3

$$\begin{aligned} \text{(P)} \quad & \min_{(x,y)} x^2 + y^2 \\ & \text{s.t. } 1 - x^4 \leq 0 \end{aligned}$$

1. Using the [desmos plot](#), what do you think the optimal solution  $f^*$  is?

*The minimum occurs at  $x = \pm 1$  and  $y = 0$ , and the optimal value is  $f^* = 1$ .*

*You can also get this by noting that the constraint is  $x^4 \geq 1$ , so  $|x| \geq 1$ .*

2. Now solve the problem using duality:

- (a) Write down the Lagrangian.

*The Lagrangian is*

$$L(x, y, \lambda) = x^2 + y^2 + \lambda(1 - x^4).$$

- (b) In a different Desmos plot from above, plot the Lagrangian as a function of  $x$  and  $y$  using a slider bar for different values of  $\lambda \geq 0$ . What do you observe? What does this suggest about the dual function  $q(\lambda) = \min_{x,y} L(x, y, \lambda)$ ? What is  $q^* = \max_{\lambda} q(\lambda)$ ?

*For an additional challenge, can you justify your observation without using Desmos?*

*When  $\lambda = 0$ , the Lagrangian is just the objective function, which is a bowl-shaped function with minimum at  $x = 0$  and  $y = 0$ .*

*When  $\lambda > 0$ , we can see that the function goes to  $-\infty$  as  $|x|$  goes to  $\infty$ , so there is no finite minimum.*

*Since  $q(\lambda) = \min_{x,y} L(x, y, \lambda)$ , we have  $q(0) = 0$  and  $q(\lambda) = -\infty$  for  $\lambda > 0$ . This means that  $q^* = \max_{\lambda} q(\lambda) = 0$ .*

3. Based on what you found above, is the bound from the weak duality theorem tight for this problem? That is, do we have  $q^* = f^*$  in this problem?

*No, we have  $q^* = 0 < 1 = f^*$ , so the bound is not tight.*