Ch 10.2: Multi-Layer Neural Nets

Lecture 30 - CMSE 381

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Announcements

Last time:

- Single Layer Neural Nets

This lecture:

- Multi-layer Neural Nets
- Application to MNIST

Announcements:

- HW 9 Due Sunday 4/13
- Exam 3 4/21
- Project Due 4/25

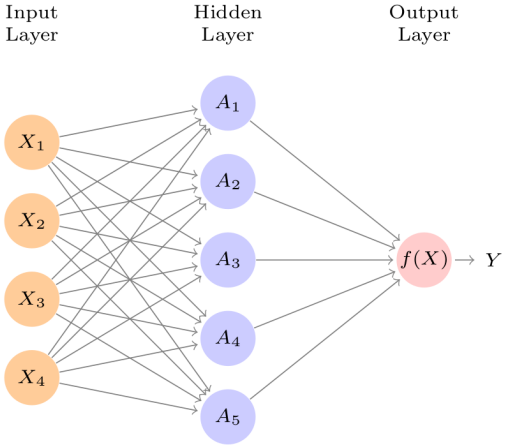
The end is near!

28	F	4/4	SVM	9.3, 9.4	HW #8 Due Sun 4/6	
29	M	4/7	Single Layer NN	10.1		Q9
30	W	4/9	Multi Layer NN	10.2		
31	F	4/11	CNN	10.3	HW #9 Due Sun 4/13	
32	M	4/14	Unsupervised learning / clustering	12.1, 12.4		Q10
33	W	4/16	Virtual: Project Office Hours			
	F	4/18	***Review***			
	M	4/21	**Midterm #3**			
	W	4/23				
	F	4/25			**Project Due**	
			No final exam			

Section 1

Neural Nets

Feed Forward Neural Network: The cartoon



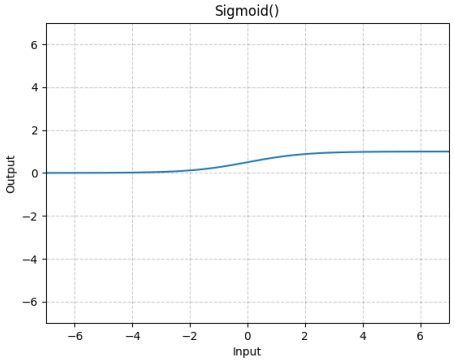
$$A_k = h_k(X) = g(w_{k0} + \sum_{j=1}^{p} w_{kj} X_j),$$

$$f(X) = \beta_0 + \sum_{k=1}^{K} \beta_k A_k$$

Choices for activation function

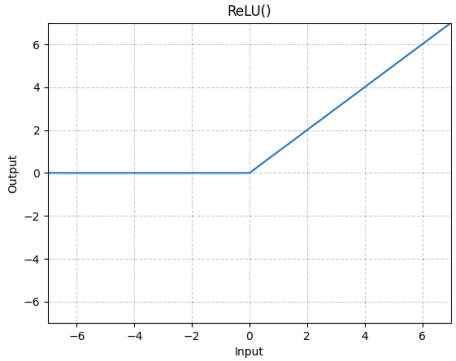
Sigmoid:

$$g(z) = \frac{e^z}{1+e^z} = \frac{1}{1+e^{-z}}$$

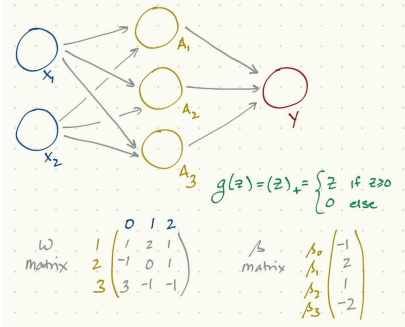


ReLU: Rectified linear unit

$$g(z) = (z)_+ = \begin{cases} 0 & \text{if } z < 0 \\ z & \text{else.} \end{cases}$$



Matrix version



$$A_k = h_k(X) = g(w_{k0} + \sum_{j=1}^{p} w_{kj} X_j),$$

$$A = g(\mathbf{W} \cdot \mathbf{X}) \qquad \mathbf{X}^T = (1\ X_1\ X_2\ \cdots\ X_p)$$

$$f(X) = \beta_0 + \sum_{k=1}^{K} \beta_k A_k$$

$$Y = \beta \cdot \mathbf{A} \qquad \mathbf{A}^T = (1\ A_1\ A_2\ \cdots\ A_K)$$

Training the model

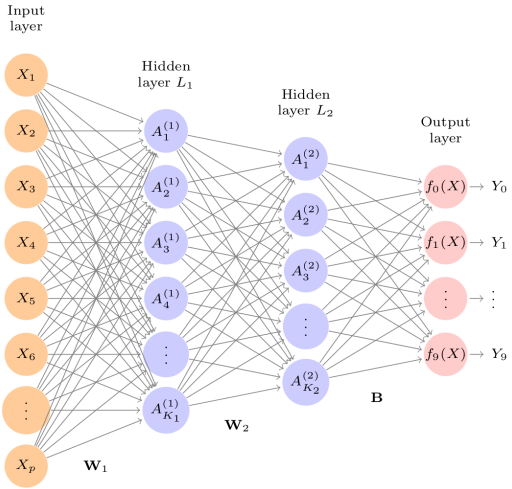
Choose parameters by minimizing RSS, $\sum_{i=1}^{n}(y_i - f(x_i))^2$ (or other loss function)

Chosen in advance: **Tuned by the model:**

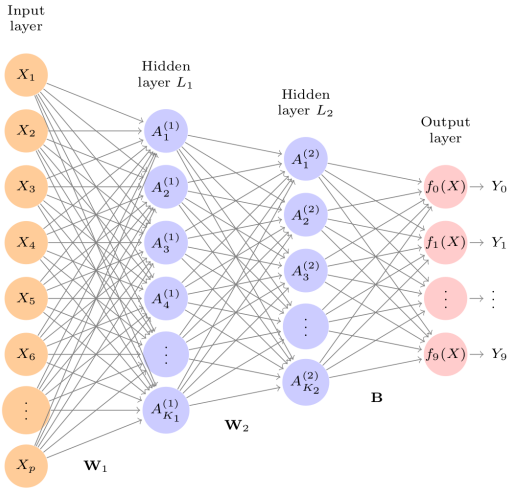
Section 2

Multilayer Neural Networks

Multiple layers

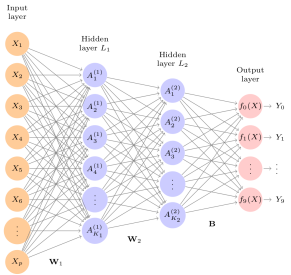


Hidden layers



$$
\begin{aligned}
A_k^{(1)} &= h_k^{(1)}(X) \\
&= g(w_{k0}^{(1)} + \textstyle\sum_{j=1}^{p} w_{kj}^{(1)} X_j) \\
A_\ell^{(2)} &= h_\ell^{(2)}(X) \\
&= g(w_{\ell 0}^{(2)} + \textstyle\sum_{k=1}^{K_1} w_{\ell k}^{(2)} A_k^{(1)})
\end{aligned}
$$

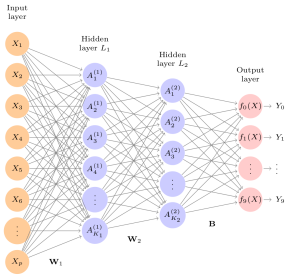
More on that architecture



$$
\begin{aligned}
A_k^{(1)} &= h_k^{(1)}(X) \\
&= g(w_{k0}^{(1)} + \textstyle\sum_{j=1}^{p} w_{kj}^{(1)} X_j)
\end{aligned}
$$

$$
\begin{aligned}
A_\ell^{(2)} &= h_\ell^{(2)}(X) \\
&= g(w_{\ell 0}^{(2)} + \textstyle\sum_{k=1}^{K_1} w_{\ell k}^{(2)} A_k^{(1)})
\end{aligned}
$$

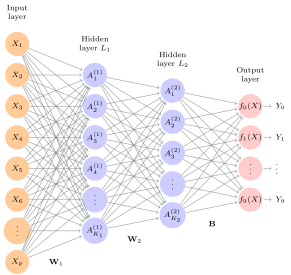
Matrix version: First layer



$$
\begin{aligned}
A_k^{(1)} &= h_k^{(1)}(X) \\
&= g(w_{k0}^{(1)} + \textstyle\sum_{j=1}^{p} w_{kj}^{(1)} X_j)
\end{aligned}
$$

$$A^{(1)} = g(\mathbf{W}^{(1)} \cdot \mathbf{X}) \qquad \mathbf{X}^T = (1\ X_1\ X_2\ \cdots\ X_p)$$

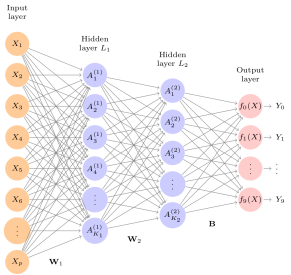
Matrix version: Second layer



$$
\begin{aligned}
A_\ell^{(2)} &= h_\ell^{(2)}(X) \\
&= g(w_{\ell 0}^{(2)} + \textstyle\sum_{k=1}^{K_1} w_{\ell k}^{(2)} A_k^{(1)})
\end{aligned}
$$

$$
A^{(2)} = g(\mathbf{W}^{(2)} \cdot \mathbf{A}) \qquad (\mathbf{A}^{(1)})^T = (1\ A_1^{(1)}\ A_2^{(1)}\ \cdots\ A_{K_1}^{(1)})
$$

Matrix version: Last layer, first step

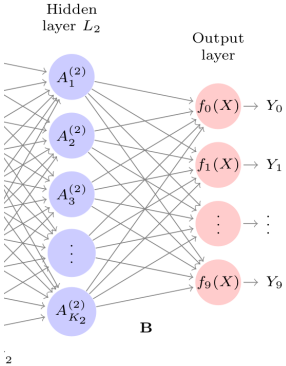


$$
\begin{aligned}
Z_m &= \beta_{m0} + \textstyle\sum_{\ell=1}^{K_2} \beta_{m\ell} h_\ell^{(2)}(X) \\
&= \beta_{m0} + \textstyle\sum_{\ell=1}^{K_2} \beta_{m\ell} A_\ell^{(2)},
\end{aligned}
$$

$$
\mathbf{Z} = \beta \cdot \mathbf{A}
$$

β is $M \times (K_2+1)$ matrix $\qquad (\mathbf{A}^{(2)})^T = (1\ A_1^{(2)}\ A_2^{(2)}\ \cdots\ A_{K_2}^{(2)})$

The last column for classification: Softmax



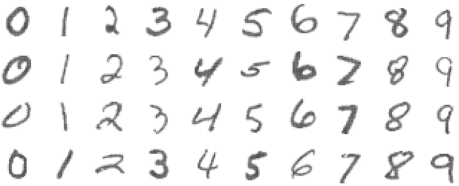
$$f_m(X) = \Pr(Y = m|X) = \frac{e^{Z_m}}{\sum_{\ell=0}^{9} e^{Z_\ell}},$$

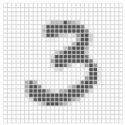
An example

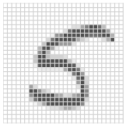
$$f_m(X) = \Pr(Y = m|X) = \frac{e^{Z_m}}{\sum_{\ell=0}^{9} e^{Z_\ell}},$$

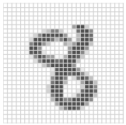
$$Z = \begin{pmatrix} 1 & 3 & -1 & 2 & 5 \end{pmatrix}$$

MNIST

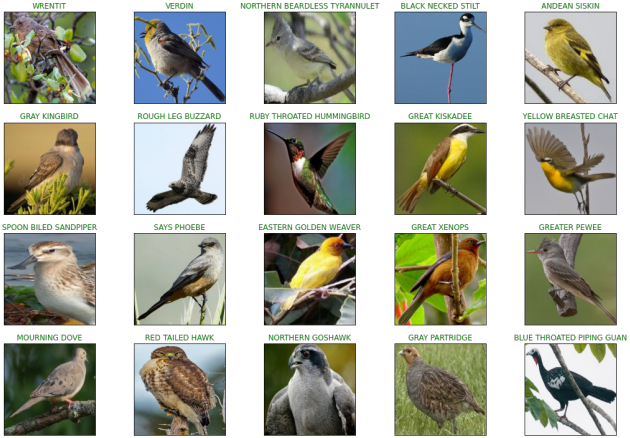








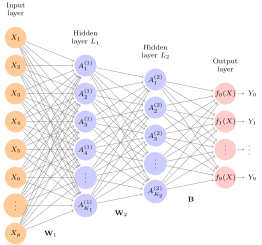
Classification



Coding

TL;DR

Feed Forward Neural Net



$$A_k = h_k(X) = g(w_{k0} + \sum_{j=1}^{p} w_{kj} X_j),$$

- Combines input data using learned weights
- Linear combo of those to get output
- Sometimes softmax to get probability of classification

Next time

21	W	3/19	Polynomial & Step Functions	7.1-7.2		
22	F	3/21	Step Functions; Basis functions; Start Splines	7.2-7.4		
23	M	3/24	Regression Splines	7.4		
24	W	3/26	Decision Trees	8.1	HW #6 Due Wed 3/26	Q7
25	F	3/28	Random Forests	8.2.1, 8.2.2	HW #7 Due Sun 3/30	
26	M	3/31	Maximal Margin Classifier	9.1		
27	W	4/2	SVC	9.2		Q8
28	F	4/4	SVM	9.3, 9.4	HW #8 Due Sun 4/6	
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