

Ch 7.2-7.4: Step Functions, Basis Functions, Start Splines

Lecture 22 - CMSE 381

Prof. Lianzhang Bao

Michigan State University

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Dept of Computational Mathematics, Science & Engineering

Fri, Mar 21, 2025

Last time:

- 7.1 Polynomial regression
- 7.2 Step functions

This lecture:

- 7.2 Step functions
- 7.3 Basis functions
- 7.4 Regression Splines (Finish next lecture)

Announcements:

- HW #6 Due Sunday
- Projects

Section 1

Last time

Polynomial regression

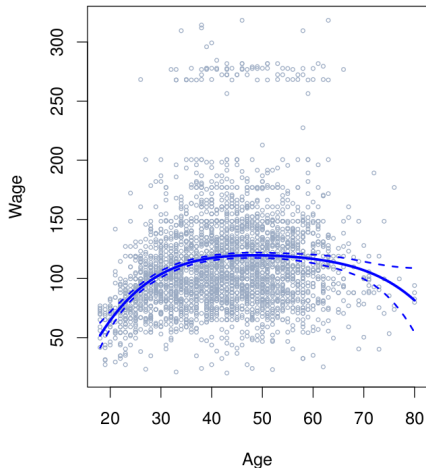
Replace linear model

$$y_i = \beta_0 + \beta_1 x_1 + \varepsilon_i$$

with

$$y_i = \beta_0 + \beta_1 x_1 + \beta_2 x_i^2 + \cdots + \beta_d x_i^d + \varepsilon_i$$

Example with wage data



$$-184.1542 + 21.24552 * age + -0.56386 * age^2 + 0.00681 * age^3 + -3e - 05 * age^4$$

Step functions

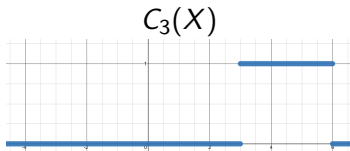
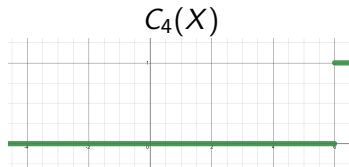
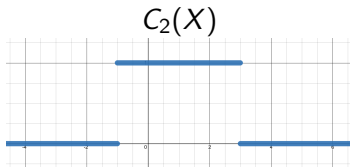
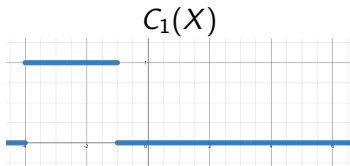
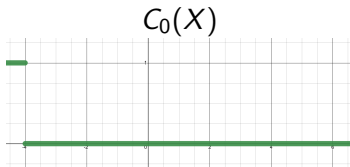
- $I(X < c)$
- $I(c_1 \leq X < c_2)$
- $I(c \leq X)$

$$\begin{aligned}C_0(X) &= I(X < c_1), \\C_1(X) &= I(c_1 \leq X < c_2), \\C_2(X) &= I(c_2 \leq X < c_3), \\&\vdots \\C_{K-1}(X) &= I(c_{K-1} \leq X < c_K), \\C_K(X) &= I(c_K \leq X),\end{aligned}$$

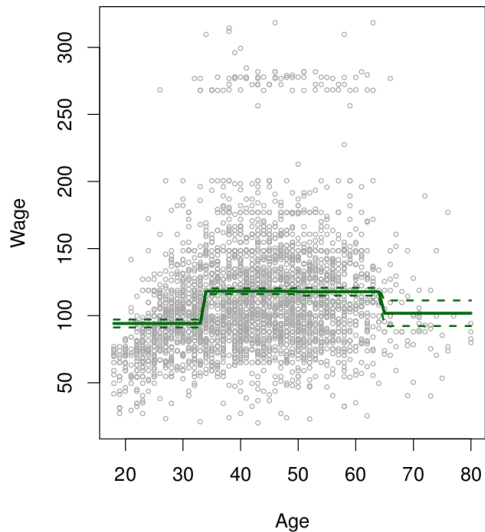
Learned model:

$$y_i = \beta_0 + \beta_1 C_1(x_i) + \beta_2 C_2(x_i) + \cdots + \beta_K C_K(x_i) + \varepsilon_i$$

Example: Cut points at -4, -1, 3, 6



Step function example

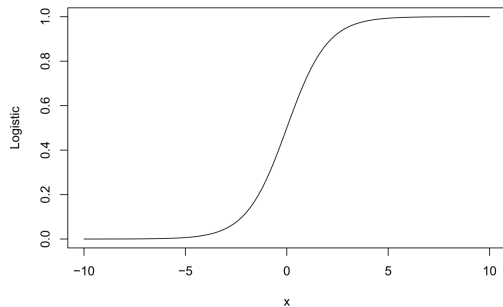


Section 2

Classification versions

Remember logistic regression?

$$y = \frac{e^x}{1 + e^x}$$



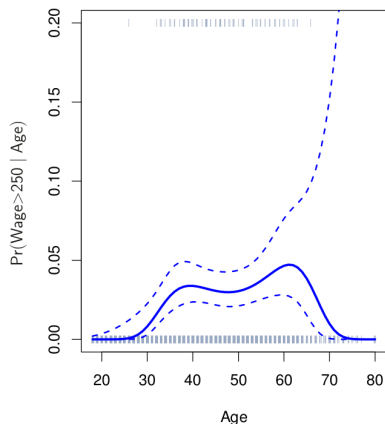
$$p(X) = \frac{e^{\beta_0 + \beta_1 X}}{1 + e^{\beta_0 + \beta_1 X}}$$

Multiple features:

$$p(X) = \frac{e^{\beta_0 + \beta_1 X_1 + \dots + \beta_p X_p}}{1 + e^{\beta_0 + \beta_1 X_1 + \dots + \beta_p X_p}}$$

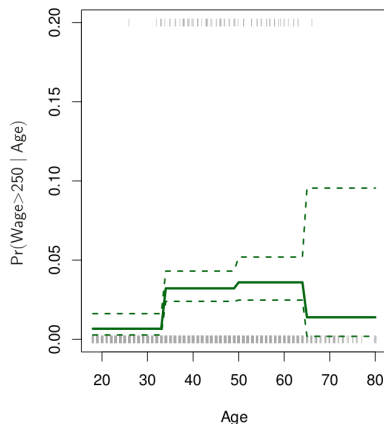
Classification version: Polynomial regression

$$\Pr(y_i > 250 \mid x_i) = \frac{\exp(\beta_0 + \beta_1 x_i + \cdots + \beta_d x_i^d)}{1 + \exp(\beta_0 + \beta_1 x_i + \cdots + \beta_d x_i^d)}$$

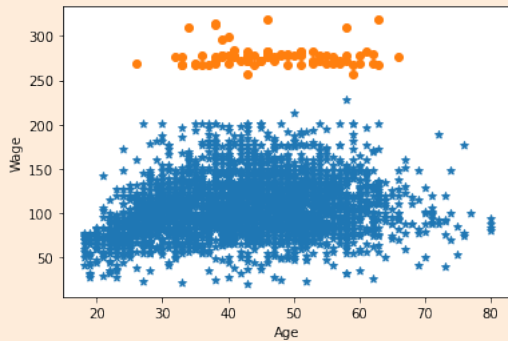


Classification version: Step functions

$$\Pr(y_i > 250 \mid x_i) = \frac{\exp(\beta_0 + \beta_1 C_1(x_i) + \beta_2 C_2(x_i) + \cdots + \beta_K C_K(x_i))}{1 + \exp(\beta_0 + \beta_1 C_1(x_i) + \beta_2 C_2(x_i) + \cdots + \beta_K C_K(x_i))}$$



Coding bit: classification version



A few more comments on step functions

- Gives the chance to break up the domain, avoid forcing global structure
- Need to make decisions about the c_i .
A bit arbitrary unless your data has natural breakpoints.
- Popular in biostats and epidemiology

Section 3

Basis functions

Basis Functions Setup

Polynomial and piecewise-constant regression models are special cases of a *basis function* approach.

$$y_i = \beta_0 + \beta_1 b_1(x_i) + \beta_2 b_2(x_i) + \cdots + \beta_K b_K(x_i) + \varepsilon_i$$

Section 4

Regression Splines

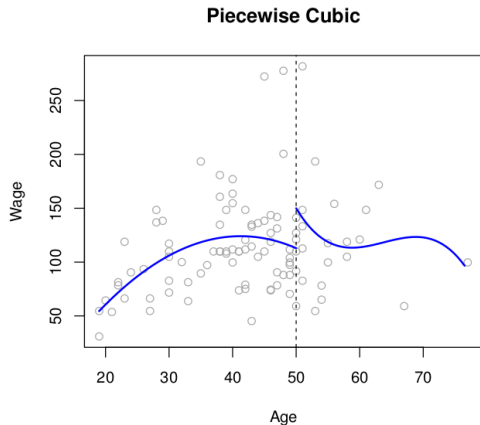
Piecewise polynomials

- Fit a polynomial regression

$$y_i = \beta_0 + \beta_1 x_1 + \beta_2 x_i^2 + \cdots + \beta_d x_i^d + \varepsilon_i$$

- Let the β_i 's be different at different locations of the range.

Example of piecewise polynomial

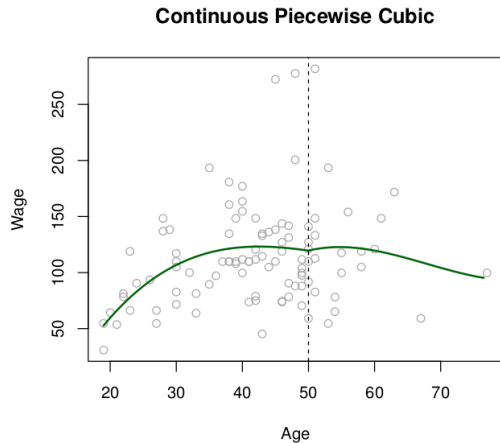


Example:

$$y_i = \begin{cases} \beta_{01} + \beta_{11}x_i + \beta_{21}x_i^2 + \beta_{31}x_i^3 + \epsilon_i & \text{if } x_i < c \\ \beta_{02} + \beta_{12}x_i + \beta_{22}x_i^2 + \beta_{32}x_i^3 + \epsilon_i & \text{if } x_i \geq c. \end{cases}$$

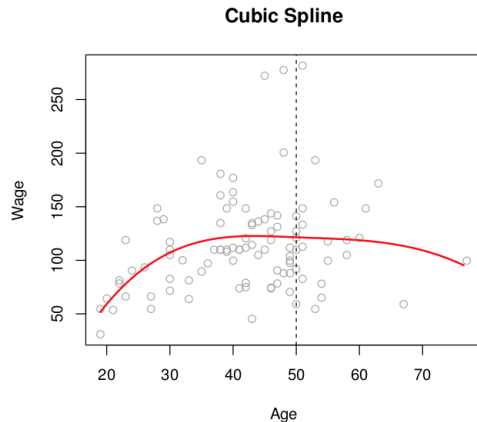
The fix

- Fit piecewise polynomial
- Require continuity at knots



The better fix: Cubic splines

- Fit piecewise polynomial
- Require continuity at knots
- Require the first and second derivatives to be continuous at knots

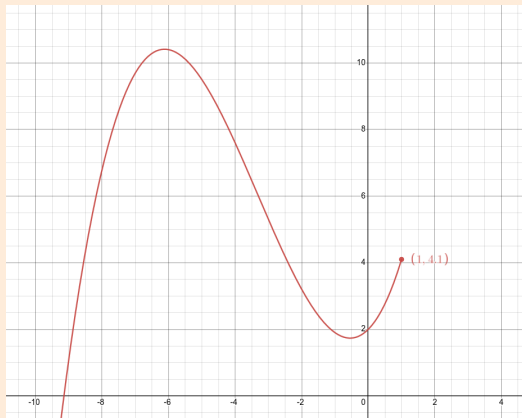


Example

We have the following piecewise cubic polynomial:

$$f(x) = \begin{cases} 2 + x + x^2 + 0.1x^3 & x \leq 1 \\ b_0 + b_1x + b_2x^2 - x^3 & x > 1 \end{cases}$$

What are b_0 , b_1 , and b_2 to make this a cubic spline?



Check your answers: [desmos.com/calculator/ns4tr7mw0n](https://www.desmos.com/calculator/ns4tr7mw0n)

More space for work

$$f(x) = \begin{cases} 2 + x + x^2 + 0.1x^3 & x \leq 1 \\ b_0 + b_1x + b_2x^2 - x^3 & x > 1 \end{cases}$$

Next time

21	W	3/19	Polynomial & Step Functions	7.1-7.2		
22	F	3/21	Step Functions; Basis functions; Start Splines	7.2-7.4	HW #6 Due Sun 3/23	Q7
23	M	3/24	Regression Splines	7.4		
24	W	3/26	Decision Trees	8.1		
25	F	3/28	Random Forests	8.2.1, 8.2.2	HW #7 Due Sun 3/30	
26	M	3/31	Maximal Margin Classifier	9.1		
27	W	4/2	SVC	9.2		Q8
28	F	4/4	SVM	9.3, 9.4	HW #8 Due Sun 4/6	
29	M	4/7	Single Layer NN	10.1		
30	W	4/9	Multi Layer NN	10.2		Q9
31	F	4/11	CNN	10.3	HW #9 Due Sun 4/13	
32	M	4/14	Unsupervised learning / clustering	12.1, 12.4		
33	W	4/16	Virtual: Project Office Hours			Q10
	F	4/18	Review			
	M	4/21	Midterm #3			
	W	4/23				
	F	4/25			Project Due	
			No final exam			