Ch 6.2: Shrinkage - The Lasso

Lecture 18 - CMSE 381

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Announcements

Last time:

- Ridge Regression

This time:

- The Lasso

Announcements:

- HW # 4 due Sunday
- Spring Break!

12	F	2/14	Leave one out CV	5.1.1, 5.1.2		
13	M	2/17	k-fold CV	5.1.3		
14	W	2/19	More k-fold CV	5.1.4-5		Q5
15	F	2/21	k-fold CV for classification	5.1.5		
16	M	2/24	Subset selection	6.1		
17	W	2/26	Shrinkage: Ridge	6.2.1		
18	F	2/28	Shrinkage: Lasso	6.2.2	HW #4 Due Sun 3/2	
	M	3/3	Spring Break			
	W	3/5	Spring Break			
	F	3/7	Spring Break			
19	M	3/10	PCA	6.3		
20	W	3/12	PCR	6.3		Q6
	F	3/14	***Review***		HW #5 Due Sun 3/16	
	M	3/17	**Midterm #2**			

Section 1

Last time

Goal

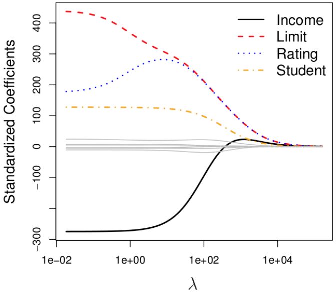
- Fit model using all p predictors
- Aim to constrain (regularize) coefficient estimates
- Shrink the coefficient estimates towards 0

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4$$

Ridge regression

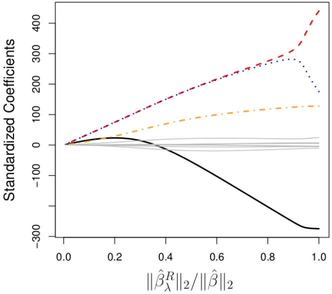
Before:

$$RSS = \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2$$



After:

$$\sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2 + \lambda \sum_{j=1}^{p} \beta_j^2 = RSS + \lambda \sum_{j=1}^{p} \beta_j^2$$



Scale equivariance (or lack thereof)

Scale equivariant: Multiplying a variable by c (cX_i) just returns a coefficient multiplied by $1/c$ ($1/c\beta_i$)

- Least squares is scale equivariant
- Ridge regression is not

Solution: standardize predictors

$$\widetilde{x}_{ij} = \frac{x_{ij}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_{ij} - \overline{x}_j)^2}}$$

Section 2

The Lasso

Same goal as before

- Fit model using all p predictors
- Aim to constrain (regularize) coefficient estimates
- Shrink the coefficient estimates towards 0

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4$$

The lasso

Least Squares:

$$RSS = \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2$$

Ridge:

$$\sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2 + \lambda \sum_{j=1}^{p} \beta_j^2 = RSS + \lambda \sum_{j=1}^{p} \beta_j^2$$

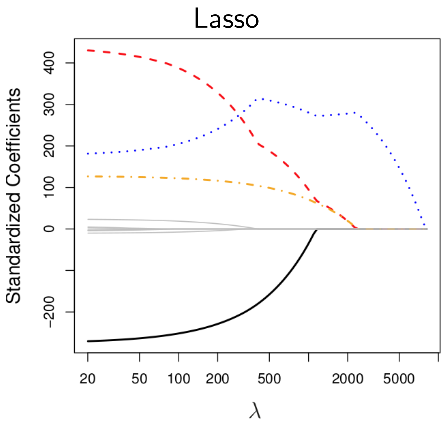
The Lasso:

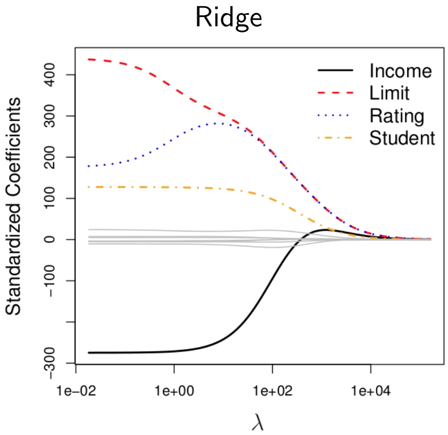
$$\sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2 + \lambda \sum_{j=1}^{p} |\beta_j| = RSS + \lambda \sum_{j=1}^{p} |\beta_j|$$

Subsets with lasso

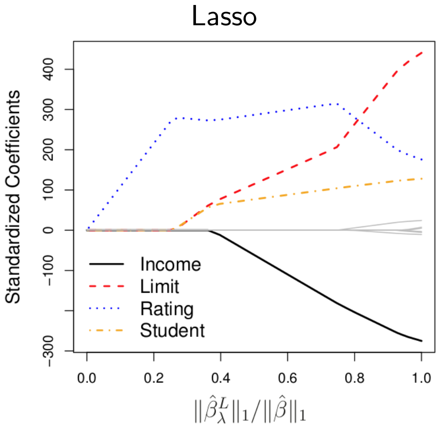
$$\sum_{i=1}^{n}\left(y_i - \beta_0 - \sum_{j=1}^{p}\beta_j x_{ij}\right)^2 + \lambda\sum_{j=1}^{p}|\beta_j| = RSS + \lambda\sum_{j=1}^{p}|\beta_j|$$

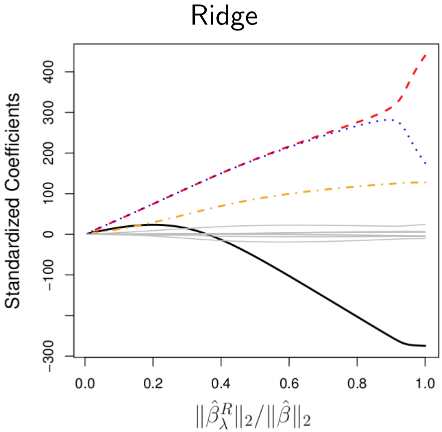
An example on `Credit` data set





More example on `Credit` data set





Scale equivavariance (or lack thereof)

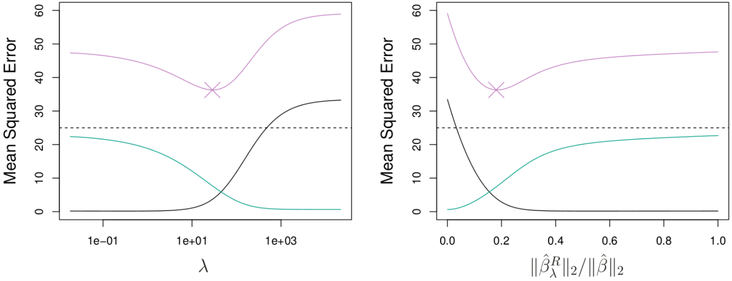
Scale equivariant: Multiplying a variable by c just returns a coefficient multiplied by $1/c$

Least squares **is** scale equivariant.
Ridge/Lasso **are very much not**.

Solution: standardize predictors

$$\widetilde{x}_{ij} = \frac{x_{ij}}{\sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_{ij} - \overline{x}_j)^2}}$$

Bias-Variance tradeoff

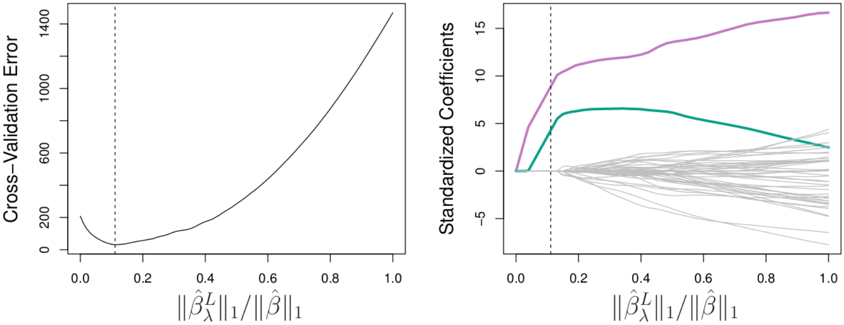


Squared bias (black), variance (green), and test mean squared error (purple) for simulated data.

Using Cross-Validation to find λ

- Choose a grid of λ values
- Compute the (k-fold) cross-validation error for each value of λ
- Select the tuning parameter value λ for which the CV error is smallest.
- The model is re-fit using all of the available observations and the selected value of the tuning parameter.

10-fold CV choice of λ for lasso and simulated data



Coding example

Ridge vs Lasso

Ridge Regression:

Lasso:

TL;DR - Original forumlation

Least Squares:

$$RSS = \sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2$$

Ridge:

$$\sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2 + \lambda \sum_{j=1}^{p} \beta_j^2 = RSS + \lambda \sum_{j=1}^{p} \beta_j^2$$

The Lasso:

$$\sum_{i=1}^{n} \left(y_i - \beta_0 - \sum_{j=1}^{p} \beta_j x_{ij} \right)^2 + \lambda \sum_{j=1}^{p} |\beta_j| = RSS + \lambda \sum_{j=1}^{p} |\beta_j|$$

Next time

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	F	3/14	***Review***		HW #5 Due Sun 3/16	
	M	3/17	**Midterm #2**			