Ch 7.4: Cubic splines

Lecture 23 - CMSE 381

Prof. Mengsen Zhang

Michigan State University
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Dept of Computational Mathematics, Science & Engineering

Wed, Oct 29, 2025

Announcements

Last time:

- 7.2 Step functions
- 7.3 Basis functions

This lecture:

- 7.4 Cubic splines

Announcements:

- Homework # 6 is due Sunday.

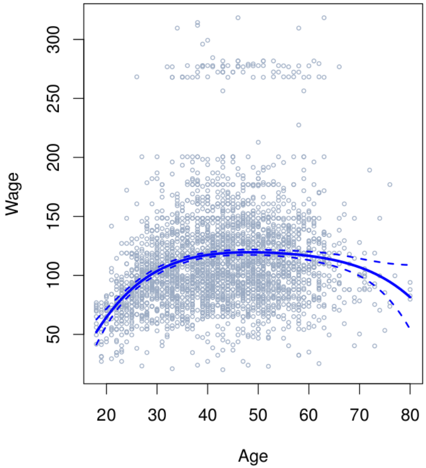
21	F	10/24	Polynomial & Step Functions	7.1-7.2	HW #5 Due Sun 10/26
22	M	10/27	Step Functions; Basis functions; Start Splines	7.2-7.4	
23	W	10/29	Regression Splines	7.4	
24	F	10/31	Decision Trees	8.1	HW #6 Due Sun 11/2
25	M	11/3	Random Forests	8.2.1, 8.2.2	
26	W	11/5	Maximal Margin Classifier	9.1	
27	F	11/7	SVC	9.2	HW #7 Due Sun 11/9
28	M	11/10	SVM	9.3, 9.4	
29	W	11/12	Single Layer NN	10.1	
30	F	11/13	Multi Layer NN	10.2	HW #8 Due Sun 11/16
31	M	11/17	CNN	10.3	
32	W	11/19	Unsupervised learning / clustering	12.1, 12.4	
33	F	11/21	***Review***		HW #9 Due Sun 11/23
	M	11/24	**Midterm #3**		
	W	11/26	Virtual: Project Office Hours		
	F	11/28	Thanksgiving		
	M	12/1	Virtual: Project Office Hours		
	W	12/3	Virtual: Project Office Hours		
	F	12/5			**Project Due**
	M	12/8			
	W	12/10			
	F	12/12	**No final exam**		**Honors Project Due**

Section 1

Previously

Polynomial regression

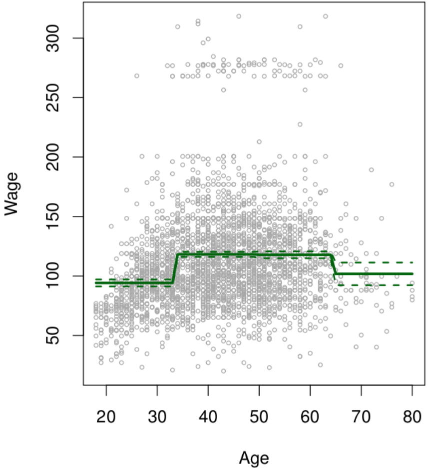
$$y_i = \beta_0 + \beta_1 x_1 + \beta_2 x_i^2 + \cdots + \beta_d x_i^d + \varepsilon_i$$



Step function regression

$$
\begin{aligned}
C_0(X) &= I(X < c_1),\\
C_1(X) &= I(c_1 \leq X < c_2),\\
C_2(X) &= I(c_2 \leq X < c_3),\\
&\vdots\\
C_{K-1}(X) &= I(c_{K-1} \leq X < c_K),\\
C_K(X) &= I(c_K \leq X),
\end{aligned}
$$

$$y_i = \beta_0 + \beta_1 C_1(x_i) + \beta_2 C_2(x_i) + \cdots + \beta_K C_K(x_i) + \varepsilon_i$$

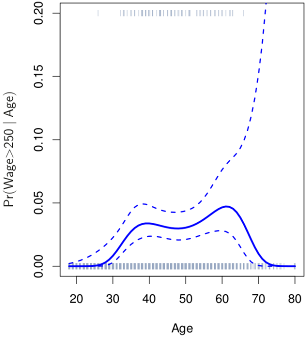


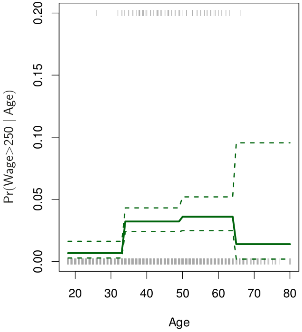
Classification version

$$\Pr(y_i > 250 \mid x_i) =$$

$$\frac{\exp(\beta_0 + \beta_1 x_i + \cdots + \beta_d x_i^d)}{1 + \exp(\beta_0 + \beta_1 x_i + \cdots + \beta_d x_i^d)}$$

$$\frac{\exp(\beta_0 + \beta_1 C_1(x_i) + \beta_2 C_2(x_i) + \cdots + \beta_K C_K(x_i))}{1 + \exp(\beta_0 + \beta_1 C_1(x_i) + \beta_2 C_2(x_i) + \cdots + \beta_K C_K(x_i))}$$





Basis Functions Setup

Polynomial and piecewise-constant regression models are special cases of a *basis function* approach.

$$y_i = \beta_0 + \beta_1 b_1(x_i) + \beta_2 b_2(x_i) + \cdots + \beta_K b_K(x_i) + \varepsilon_i$$

What will you learn today?

- How to define cubic splines mathematically in terms of piecewise polynomials? (review)
  - Write the equations for the model on each interval between knots.
  - Write the continuity constraints (function, first and second derivatives) at each knot.
  - Compute the total degrees of freedom.
- How to define cubic splines in terms of basis functions?
  - Describe them mathematically using the truncated power basis (textbook version).
  - Visualize them as B-spline basis functions (e.g., in Python / scikit-learn).
- How to fit a cubic spline model to data in Python?
- How to change the model flexibility?
  - What are the relevant metaparameters?
  - How to choose appropriate flexibility?
- What precautions must you take at the outer boundary? Why?

Section 2

Regression Splines

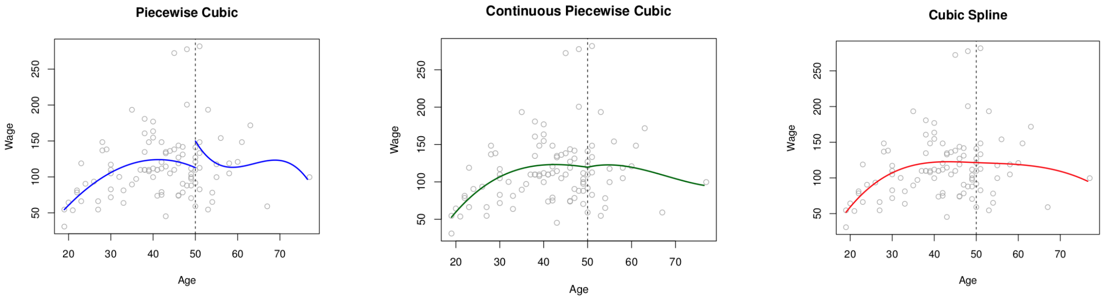
Piecewise polynomials

- Fit a polynomial regression

  $$y_i = \beta_0 + \beta_1 x_1 + \beta_2 x_i^2 + \cdots + \beta_d x_i^d + \varepsilon_i$$

- Let the β_i's be different at different locations of the range.

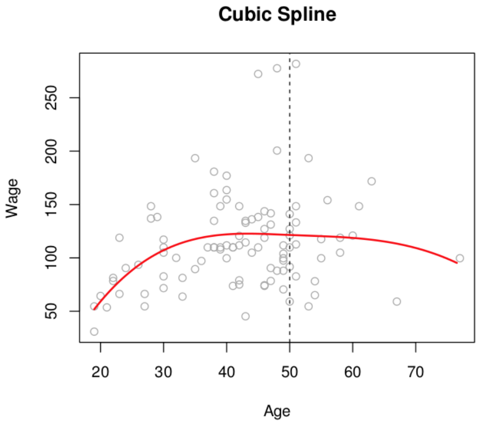
Building up to cubic splines



$$y_i = \begin{cases} \beta_{01} + \beta_{11}x_i + \beta_{21}x_i^2 + \beta_{31}x_i^3 + \epsilon_i & \text{if } x_i < c \\ \beta_{02} + \beta_{12}x_i + \beta_{22}x_i^2 + \beta_{32}x_i^3 + \epsilon_i & \text{if } x_i \geq c. \end{cases}$$

Cubic splines: degrees of freedom

$$f(x) = \begin{cases} \beta_0^1 + \beta_1^1 x + \beta_2^1 x^2 + \beta_3^1 x^3 & x < c \\ \beta_0^2 + \beta_1^2 x + \beta_2^2 x^2 + \beta_3^2 x^3 & x > c \end{cases}$$



Spline basis representation

Want to pick b_i so that we represent a cubic spline with K knots as

$$y_i = \beta_0 + \beta_1 b_1(x_i) + \beta_2 b_2(x_i) + \cdots + \beta_{K+3} b_{K+3}(x_i) + \varepsilon_i$$

Version 1: Truncated power basis function

$$h(x, z) = (x - z)^3_+ = \begin{cases} (x - z)^3 & \text{if } x > z \\ 0 & \textit{else} \end{cases}$$

Desmos link: https://www.desmos.com/calculator/ahllu5glar

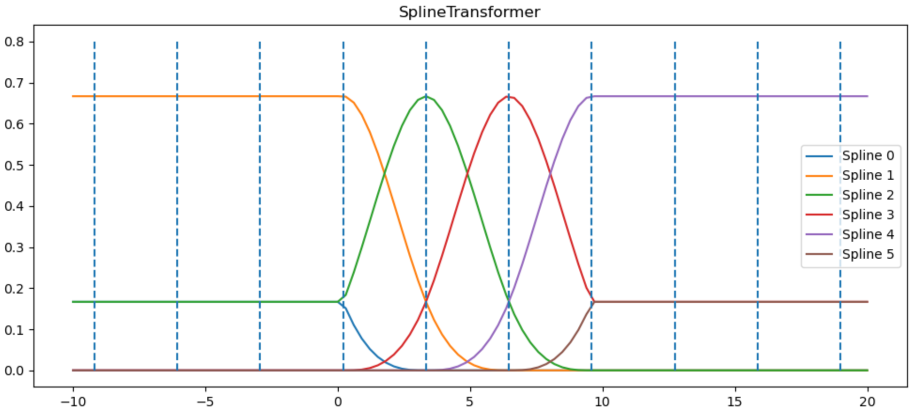
The (book) basis for cubic splines

Given knots at $z_1, \cdots, z_K$

- X
- X^2
- X^3
- $h(X, z_1)$
- $h(X, z_2)$
- $\vdots$
- $h(X, z_K)$

$$f(X) = \beta_0 + \beta_1 X + \beta_2 X^2 + \beta_3 X^3 + \beta_4 h(X, z_1) + \beta_5 h(X, z_2) + \cdots + \beta_{k+3} h(X, z_K)$$

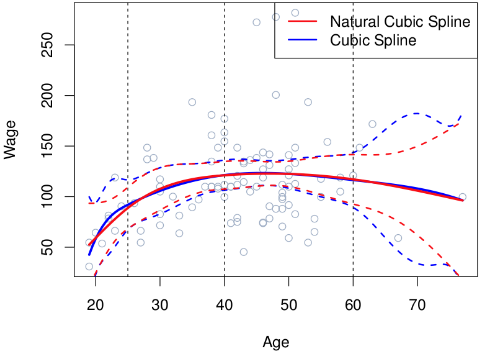
Version 2: B-spline basis function



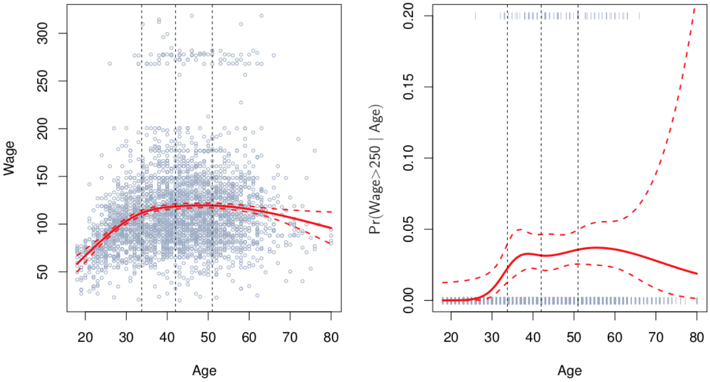
Test your understanding: PollEv

Coding example

Notes on cubic splines

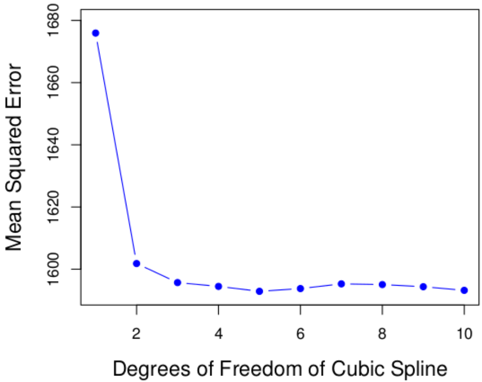


Where to put the knots?

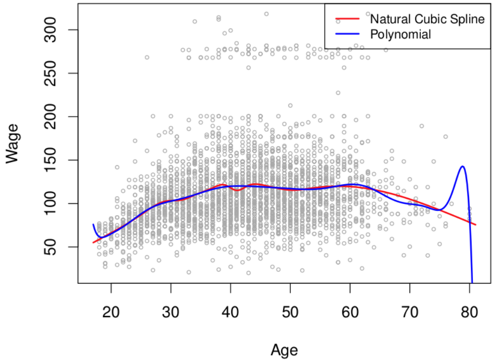


How many knots to use?

When in doubt, Cross-Validate.



Cubic splines vs Polynomial Regression



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